Chapter 5

Determinants

5.1 The Properties of Determinants

The determinant of a square matrix is a single number. That number contains an amazing amount of information about the matrix. It tells immediately whether the matrix is invertible. ***The determinant is zero when the matrix has no inverse.*** When A is invertible, the determinant of A^{-1} is $1/(\det A)$. If $\det A = 2$ then $\det A^{-1} = \frac{1}{2}$. In fact the determinant leads to a formula for every entry in A^{-1}.

This is one use for determinants—to find formulas for inverse matrices and pivots and solutions $A^{-1}\boldsymbol{b}$. For a large matrix we seldom use those formulas, because elimination is faster. For a 2 by 2 matrix with entries a, b, c, d, its determinant $ad - bc$ shows how A^{-1} changes as A changes:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \text{has inverse} \quad A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}. \tag{1}$$

Multiply those matrices to get I. When the determinant is $ad - bc = 0$, we are asked to divide by zero and we can't—then A has no inverse. (The rows are parallel when $a/c = b/d$. This gives $ad = bc$ and $\det A = 0$). Dependent rows always lead to $\det A = 0$.

The determinant is also connected to the pivots. For a 2 by 2 matrix the pivots are a and $d - (c/a)b$. ***The product of the pivots is the determinant***:

$$\textbf{Product of pivots} \qquad a\left(d - \frac{c}{a}b\right) = ad - bc \quad \textbf{which is} \quad \det A.$$

After a row exchange the pivots change to c and $b - (a/c)d$. Those new pivots multiply to give $bc - ad$. The row exchange to $\left[\begin{smallmatrix} c & d \\ a & b \end{smallmatrix}\right]$ reversed the sign of the determinant.

Looking ahead The determinant of an n by n matrix can be found in three ways:

1 Multiply the n pivots (times 1 or -1)	This is the **pivot formula.**
2 Add up $n!$ terms (times 1 or -1)	This is the **"big" formula.**
3 Combine n smaller determinants (times 1 or -1)	This is the **cofactor formula.**

You see that *plus or minus signs*—the decisions between 1 and -1—play a big part in determinants. That comes from the following rule for n by n matrices:

The determinant changes sign when two rows* (or two columns) *are exchanged.

The identity matrix has determinant $+1$. Exchange two rows and $\det P = -1$. Exchange two more rows and the new permutation has $\det P = +1$. Half of all permutations are *even* ($\det P = 1$) and half are *odd* ($\det P = -1$). Starting from I, half of the P's involve an even number of exchanges and half require an odd number. In the 2 by 2 case, ad has a plus sign and bc has minus—coming from the row exchange:

$$\det\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} = 1 \quad\text{and}\quad \det\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix} = -1.$$

The other essential rule is linearity—but a warning comes first. Linearity does not mean that $\det(A+B) = \det A + \det B$. ***This is absolutely false.*** That kind of linearity is not even true when $A = I$ and $B = I$. The false rule would say that $\det(I+I) = 1+1 = 2$. The true rule is $\det 2I = 2^n$. Determinants are multiplied by 2^n (not just by 2) when matrices are multiplied by 2.

We don't intend to define the determinant by its formulas. It is better to start with its properties—*sign reversal and linearity*. The properties are simple (Section 5.1). They prepare for the formulas (Section 5.2). Then come the applications, including these three:

(1) Determinants give A^{-1} and $A^{-1}b$ (this formula is called **Cramer's Rule**).

(2) When the edges of a box are the rows of A, the **volume** is $|\det A|$.

(3) For n special numbers λ, called **eigenvalues**, the determinants of $A - \lambda I$ is zero. This is a truly important application and it fills Chapter 6.

The Properties of the Determinant

Determinants have three basic properties (rules 1, 2, 3). By using those rules we can compute the determinant of any square matrix A. ***This number is written in two ways,* det *A and* |*A*|.** Notice: Brackets for the matrix, straight bars for its determinant. When A is a 2 by 2 matrix, the three properties lead to the answer we expect:

$$\text{The determinant of}\quad \begin{bmatrix}a & b\\ c & d\end{bmatrix}\quad\text{is}\quad \begin{vmatrix}a & b\\ c & d\end{vmatrix} = ad - bc.$$

The last rules are $\det(AB) = (\det A)(\det B)$ and $\det A^{\mathrm{T}} = \det A$. We will check all rules with the 2 by 2 formula, but do not forget: The rules apply to any n by n matrix. We will show how rules $4-10$ always follow from $1-3$.

Rule 1 (the easiest) matches $\det I = 1$ with the volume $= 1$ for a unit cube.

1 ***The determinant of the n by n identity matrix is* 1.**

$$\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \quad \text{and} \quad \begin{vmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{vmatrix} = 1.$$

2 ***The determinant changes sign when two rows are exchanged*** (sign reversal):

$$\text{Check:} \quad \begin{vmatrix} c & d \\ a & b \end{vmatrix} = - \begin{vmatrix} a & b \\ c & d \end{vmatrix} \quad \text{(both sides equal } bc - ad\text{).}$$

Because of this rule, we can find $\det P$ for any permutation matrix. Just exchange rows of I until you reach P. Then $\det P = +1$ for an ***even*** number of row exchanges and $\det P = -1$ for an ***odd*** number.

The third rule has to make the big jump to the determinants of all matrices.

3 ***The determinant is a linear function of each row separately*** (all other rows stay fixed). If the first row is multiplied by t, the determinant is multiplied by t. If first rows are added, determinants are added. This rule only applies when the other rows do not change! Notice how c and d stay the same:

multiply row 1 by any number t $\quad \begin{vmatrix} ta & tb \\ c & d \end{vmatrix} = t \begin{vmatrix} a & b \\ c & d \end{vmatrix}$

add row 1 of A to row 1 of A': $\quad \begin{vmatrix} a+a' & b+b' \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} a' & b' \\ c & d \end{vmatrix}.$

In the first case, both sides are $tad - tbc$. Then t factors out. In the second case, both sides are $ad + a'd - bc - b'c$. These rules still apply when A is n by n, and the last $n-1$ rows don't change. May we emphasize rule 3 with numbers:

$$\begin{vmatrix} 4 & 8 & 8 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix} = 4 \begin{vmatrix} 1 & 2 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix} \quad \text{and} \quad \begin{vmatrix} 4 & 8 & 8 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 4 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix} + \begin{vmatrix} 0 & 8 & 8 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix}.$$

By itself, rule 3 does not say what those determinants are (the first one is 4).

Combining multiplication and addition, we get any linear combination in one row (the other rows must stay the same). Any row can be the one that changes, since rule 2 for row exchanges can put it up into the first row and back again.

This rule does not mean that $\det 2I = 2 \det I$. To obtain $2I$ we have to multiply *both* rows by 2, and the factor 2 comes out both times:

$$\begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} = 2^2 = 4 \quad \text{and} \quad \begin{vmatrix} t & 0 \\ 0 & t \end{vmatrix} = t^2.$$

This is just like area and volume. Expand a rectangle by 2 and its area increases by 4. Expand an n-dimensional box by t and its volume increases by t^n. The connection is no accident—we will see how *determinants equal volumes*.

Pay special attention to rules 1–3. They completely determine the number $\det A$. We could stop here to find a formula for n by n determinants. (a little complicated) We prefer to go gradually, with other properties that follow directly from the first three. These extra rules $4 - 10$ make determinants much easier to work with.

4 ***If two rows of A are equal, then* det A = 0.**

Equal rows Check 2 by 2 : $\begin{vmatrix} a & b \\ a & b \end{vmatrix} = 0.$

Rule 4 follows from rule 2. (Remember we must use the rules and not the 2 by 2 formula.) *Exchange the two equal rows.* The determinant D is supposed to change sign. But also D has to stay the same, because the matrix is not changed. The only number with $-D = D$ is $D = 0$—this must be the determinant. (Note: In Boolean algebra the reasoning fails, because $-1 = 1$. Then D is defined by rules 1, 3, 4.)

A matrix with two equal rows has no inverse. Rule 4 makes $\det A = 0$. But matrices can be singular and determinants can be zero without having equal rows! Rule 5 will be the key. We can do row operations without changing $\det A$.

5 ***Subtracting a multiple of one row from another row leaves* det A *unchanged.***

ℓ times row 1
from row 2

$$\begin{vmatrix} a & b \\ c-\ell a & d-\ell b \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix}.$$

Rule 3 (linearity) splits the left side into the right side plus another term $-\ell\begin{vmatrix} a & b \\ a & b \end{vmatrix}$. This extra term is zero by rule 4. Therefore rule 5 is correct (not just 2 by 2).

Conclusion *The determinant is not changed by the usual elimination steps from A to U.* Thus $\det A$ *equals* $\det U$. If we can find determinants of triangular matrices U, we can find determinants of all matrices A. Every row exchange reverses the sign, so always $\det A = \pm \det U$. Rule 5 has narrowed the problem to triangular matrices.

6 ***A matrix with a row of zeros has* det A = 0.**

Row of zeros $\begin{vmatrix} 0 & 0 \\ c & d \end{vmatrix} = 0$ and $\begin{vmatrix} a & b \\ 0 & 0 \end{vmatrix} = 0.$

For an easy proof, add some other row to the zero row. The determinant is not changed (rule 5). But the matrix now has two equal rows. So $\det A = 0$ by rule 4.

7 ***If A is triangular then* det $A = a_{11}a_{22}\cdots a_{nn}$ *= product of diagonal entries.***

Triangular $\begin{vmatrix} a & b \\ 0 & d \end{vmatrix} = ad$ and also $\begin{vmatrix} a & 0 \\ c & d \end{vmatrix} = ad.$

Suppose all diagonal entries of A are nonzero. Eliminate the off-diagonal entries by the usual steps. (If A is lower triangular, subtract multiples of each row from lower rows. If A

is upper triangular, subtract from higher rows.) By rule 5 the determinant is not changed—and now the matrix is diagonal:

Diagonal matrix $$\det \begin{bmatrix} a_{11} & & & 0 \\ & a_{22} & & \\ & & \ddots & \\ 0 & & & a_{nn} \end{bmatrix} = a_{11}a_{22}\cdots a_{nn}.$$

Factor a_{11} from the first row by rule 3. Then factor a_{22} from the second row. Eventually factor a_{nn} from the last row. The determinant is a_{11} times a_{22} times $\cdots$ times a_{nn} times $\det I$. Then rule 1 (used at last!) is $\det I = 1$.

What if a diagonal entry a_{ii} is zero? Then the triangular A is singular. Elimination produces a *zero row*. By rule 5 the determinant is unchanged, and by rule 6 a zero row means $\det A = 0$. Triangular matrices have easy determinants.

8 ***If A is singular then* $\det A = 0$. *If A is invertible then* $\det A \neq 0$.**

Singular $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is singular if and only if $ad - bc = 0$.

Proof Elimination goes from A to U. If A is singular then U has a zero row. The rules give $\det A = \det U = 0$. If A is invertible then U has the pivots along its diagonal. The product of nonzero pivots (using rule 7) gives a nonzero determinant:

Multiply pivots $$\det A = \pm \det U = \pm \text{ (product of the pivots).} \tag{2}$$

The pivots of a 2 by 2 matrix (if $a \neq 0$) are a and $d - (bc/a)$:

The determinant is $$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ 0 & d - (bc/a) \end{vmatrix} = ad - bc.$$

This is the first formula for the determinant. MATLAB uses it to find $\det A$ from the pivots. The sign in $\pm \det u$ depends on whether the number of row exchanges is even or odd. In other words, $+1$ or -1 is the determinant of the permutation matrix P that exchanges rows. With no row exchanges, the number zero is even and $P = I$ and $\det A = \det U =$ *product of pivots*. Always $\det L = 1$, because L is triangular with 1's on the diagonal. What we have is this:

$$\text{If} \quad PA = LU \quad \text{then} \quad \det P \ \det A = \det L \ \det U. \tag{3}$$

Again, $\det P = \pm 1$ and $\det A = \pm \det U$. Equation (3) is our first case of rule 9.

9 ***The determinant of AB is* $\det A$ *times* $\det B$: $|AB| = |A|\,|B|$.**

Product rule $$\begin{vmatrix} a & b \\ c & d \end{vmatrix} \begin{vmatrix} p & q \\ r & s \end{vmatrix} = \begin{vmatrix} ap + br & aq + bs \\ cp + dr & cq + ds \end{vmatrix}.$$

When the matrix B is A^{-1}, *this rule says that the determinant of* A^{-1} is $\mathbf{1}/\det A$:

$$\textbf{A times } A^{-1} \qquad AA^{-1} = I \quad \text{so} \quad (\det A)(\det A^{-1}) = \det I = 1.$$

This product rule is the most intricate so far. Even the 2 by 2 case needs some algebra:

$$|A|\,|B| = (ad - bc)(ps - qr) = (ap + br)(cq + ds) - (aq + bs)(cp + dr) = |AB|.$$

For the n by n case, here is a snappy proof that $|AB| = |A|\,|B|$. When $|B|$ is not zero, consider the ratio $D(A) = |AB|/|B|$. *Check that this ratio has properties* 1,2,3. Then $D(A)$ has to be the determinant and we have $|A| = |AB|/|B|$: good.

Property 1 *(Determinant of I)* If $A = I$ then the ratio becomes $|B|/|B| = 1$.

Property 2 *(Sign reversal)* When two rows of A are exchanged, so are the same two rows of AB. Therefore $|AB|$ changes sign and so does the ratio $|AB|/|B|$.

Property 3 *(Linearity)* When row 1 of A is multiplied by t, so is row 1 of AB. This multiplies $|AB|$ by t and multiplies the ratio by t—as desired.
Add row 1 of A to row 1 of A'. Then row 1 of AB adds to row 1 of $A'B$. By rule 3, determinants add. After dividing by $|B|$, the ratios add—as desired.

Conclusion This ratio $|AB|/|B|$ has the same three properties that define $|A|$. Therefore it equals $|A|$. This proves the product rule $|AB| = |A|\,|B|$. The case $|B| = 0$ is separate and easy, because AB is singular when B is singular. Then $|AB| = |A|\,|B|$ is $0 = 0$.

10 ***The transpose A^{T} has the same determinant as A.***

$$\textbf{Transpose} \quad \begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} a & c \\ b & d \end{vmatrix} \quad \text{since both sides equal} \quad ad - bc.$$

The equation $|A^{\mathrm{T}}| = |A|$ becomes $0 = 0$ when A is singular (we know that A^{T} is also singular). Otherwise A has the usual factorization $PA = LU$. Transposing both sides gives $A^{\mathrm{T}}P^{\mathrm{T}} = U^{\mathrm{T}}L^{\mathrm{T}}$. The proof of $|A| = |A^{\mathrm{T}}|$ comes by using rule 9 for products:

$$\text{Compare} \quad \det P \det A = \det L \det U \quad \text{with} \quad \det A^{\mathrm{T}} \det P^{\mathrm{T}} = \det U^{\mathrm{T}} \det L^{\mathrm{T}}.$$

First, $\det L = \det L^{\mathrm{T}} = 1$ (both have 1's on the diagonal). Second, $\det U = \det U^{\mathrm{T}}$ (those triangular matrices have the same diagonal). Third, $\det P = \det P^{\mathrm{T}}$ (permutations have $P^{\mathrm{T}}P = I$, so $|P^{\mathrm{T}}|\,|P| = 1$ by rule 9; thus $|P|$ and $|P^{\mathrm{T}}|$ both equal 1 or both equal -1). So L, U, P have the same determinants as $L^{\mathrm{T}}, U^{\mathrm{T}}, P^{\mathrm{T}}$ and this leaves $\det A = \det A^{\mathrm{T}}$.

Important comment on columns Every rule for the rows can apply to the columns (just by transposing, since $|A| = |A^{\mathrm{T}}|$). The determinant changes sign when two columns are exchanged. *A zero column or two equal columns will make the determinant zero.* If a column is multiplied by t, so is the determinant. The determinant is a linear function of each column separately.

It is time to stop. The list of properties is long enough. Next we find and use an explicit formula for the determinant.

■ REVIEW OF THE KEY IDEAS ■

1. The determinant is defined by $\det I = 1$, sign reversal, and linearity in each row.

2. After elimination $\det A$ is $\pm$ (product of the pivots).

3. The determinant is zero exactly when A is not invertible.

4. Two remarkable properties are $\det AB = (\det A)(\det B)$ and $\det A^{\mathrm{T}} = \det A$.

■ WORKED EXAMPLES ■

5.1 A Apply these operations to A and find the determinants of M_1, M_2, M_3, M_4:
In M_1, multiplying each a_{ij} by $(-1)^{i+j}$ gives a checkerboard sign pattern.
In M_2, rows $1, 2, 3$ of A are *subtracted* from rows $2, 3, 1$.
In M_3, rows $1, 2, 3$ of A are *added* to rows $2, 3, 1$.
How are the determinants of M_1, M_2, M_3 related to the determinant of A?

$$\begin{bmatrix} a_{11} & -a_{12} & a_{13} \\ -a_{21} & a_{22} & -a_{23} \\ a_{31} & -a_{32} & a_{33} \end{bmatrix} \quad \begin{bmatrix} \text{row 1} - \text{row 3} \\ \text{row 2} - \text{row 1} \\ \text{row 3} - \text{row 2} \end{bmatrix} \quad \begin{bmatrix} \text{row 1} + \text{row 3} \\ \text{row 2} + \text{row 1} \\ \text{row 3} + \text{row 2} \end{bmatrix}$$

Solution The three determinants are $\det A$, 0, and $2\det A$. Here are reasons:

$$M_1 = \begin{bmatrix} 1 & & \\ & -1 & \\ & & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} 1 & & \\ & -1 & \\ & & 1 \end{bmatrix} \quad \text{so } \det M_1 = (-1)(\det A)(-1).$$

M_2 is singular because its rows add to the zero row. Its determinant is zero.
M_3 can be split into *eight matrices* by Rule 3 (linearity in each row seperately):

$$\begin{vmatrix} \text{row 1 + row 3} \\ \text{row 2 + row 1} \\ \text{row 3 + row 3} \end{vmatrix} = \begin{vmatrix} \text{row 1} \\ \text{row 2} \\ \text{row 3} \end{vmatrix} + \begin{vmatrix} \text{row 3} \\ \text{row 2} \\ \text{row 3} \end{vmatrix} + \begin{vmatrix} \text{row 1} \\ \text{row 1} \\ \text{row 3} \end{vmatrix} + \cdots + \begin{vmatrix} \text{row 3} \\ \text{row 1} \\ \text{row 2} \end{vmatrix}.$$

All but the first and last have repeated rows and zero determinant. The first is A and the last has *two* row exchanges. So $\det M_3 = \det A + \det A$. (Try $A = I$.)

5.1 B Explain how to reach this determinant by row operations:

$$\det\begin{bmatrix} 1-a & 1 & 1 \\ 1 & 1-a & 1 \\ 1 & 1 & 1-a \end{bmatrix} = a^2(3-a). \tag{4}$$

Solution Subtract row 3 from row 1 and then from row 2. This leaves

$$\det\begin{bmatrix} -a & 0 & a \\ 0 & -a & a \\ 1 & 1 & 1-a \end{bmatrix}.$$

Now add column 1 to column 3, and also column 2 to column 3. This leaves a lower triangular matrix with $-a, -a, 3-a$ on the diagonal: $\det = (-a)(-a)(3-a)$.

The determinant is zero if $a = 0$ or $a = 3$. For $a = 0$ we have the *all-ones matrix*—certainly singular. For $a = 3$, each row adds to zero - again singular. Those numbers 0 and 3 are the eigenvalues of the all-ones matrix. This example is revealing and important, leading toward Chapter 6.

Problem Set 5.1

Questions 1–12 are about the rules for determinants.

1 If a 4 by 4 matrix has $\det A = \frac{1}{2}$, find $\det(2A)$ and $\det(-A)$ and $\det(A^2)$ and $\det(A^{-1})$.

2 If a 3 by 3 matrix has $\det A = -1$, find $\det(\frac{1}{2}A)$ and $\det(-A)$ and $\det(A^2)$ and $\det(A^{-1})$.

3 True or false, with a reason if true or a counterexample if false:

(a) The determinant of $I + A$ is $1 + \det A$.

(b) The determinant of ABC is $|A|\,|B|\,|C|$.

(c) The determinant of $4A$ is $4|A|$.

(d) The determinant of $AB - BA$ is zero. Try an example with $A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$.

4 Which row exchanges show that these "reverse identity matrices" J_3 and J_4 have $|J_3| = -1$ but $|J_4| = +1$?

$$\det\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = -1 \quad \text{but} \quad \det\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} = +1.$$

5 For $n = 5, 6, 7$, count the row exchanges to permute the reverse identity J_n to the identity matrix I_n. Propose a rule for every size n and predict whether J_{101} has determinant $+1$ or -1.

6 Show how Rule 6 (determinant $= 0$ if a row is all zero) comes from Rule 3.

7 Find the determinants of rotations and reflections:

$$Q = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \quad \text{and} \quad Q = \begin{bmatrix} 1-2\cos^2\theta & -2\cos\theta\sin\theta \\ -2\cos\theta\sin\theta & 1-2\sin^2\theta \end{bmatrix}.$$

8 Prove that every orthogonal matrix ($Q^{\mathrm{T}}Q = I$) has determinant 1 or -1.

(a) Use the product rule $|AB| = |A|\,|B|$ and the transpose rule $|Q| = |Q^{\mathrm{T}}|$.

(b) Use only the product rule. If $|\det Q| > 1$ then $\det Q^n = (\det Q)^n$ blows up. How do you know this can't happen to Q^n?

9 Do these matrices have determinant 0, 1, 2, or 3?

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

10 If the entries in every row of A add to zero, solve $Ax = \mathbf{0}$ to prove $\det A = 0$. If those entries add to one, show that $\det(A - I) = 0$. Does this mean $\det A = 1$?

11 Suppose that $CD = -DC$ and find the flaw in this reasoning: Taking determinants gives $|C|\,|D| = -|D|\,|C|$. Therefore $|C| = 0$ or $|D| = 0$. One or both of the matrices must be singular. (That is not true.)

12 The inverse of a 2 by 2 matrix seems to have determinant $= 1$:

$$\det A^{-1} = \det \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{ad-bc}{ad-bc} = 1.$$

What is wrong with this calculation? What is the correct $\det A^{-1}$?

Questions 13–27 use the rules to compute specific determinants.

13 Reduce A to U and find $\det A$ = product of the pivots:

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix} \qquad A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 2 & 3 \\ 3 & 3 & 3 \end{bmatrix}.$$

14 By applying row operations to produce an upper triangular U, compute

$$\det\begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 6 & 6 & 1 \\ -1 & 0 & 0 & 3 \\ 0 & 2 & 0 & 7 \end{bmatrix} \quad \text{and} \quad \det\begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}.$$

15 Use row operations to simplify and compute these determinants:

$$\det\begin{bmatrix}101 & 201 & 301\\ 102 & 202 & 302\\ 103 & 203 & 303\end{bmatrix} \quad\text{and}\quad \det\begin{bmatrix}1 & t & t^2\\ t & 1 & t\\ t^2 & t & 1\end{bmatrix}.$$

16 Find the determinants of a rank one matrix and a skew-symmetric matrix:

$$A = \begin{bmatrix}1\\ 2\\ 3\end{bmatrix}\begin{bmatrix}1 & -4 & 5\end{bmatrix} \quad\text{and}\quad K = \begin{bmatrix}0 & 1 & 3\\ -1 & 0 & 4\\ -3 & -4 & 0\end{bmatrix}.$$

17 A skew-symmetric matrix has $K^{\mathrm{T}} = -K$. Insert a, b, c for $1, 3, 4$ in Question 16 and show that $|K| = 0$. Write down a 4 by 4 example with $|K| = 1$.

18 Use row operations to show that the 3 by 3 "Vandermonde determinant" is

$$\det\begin{bmatrix}1 & a & a^2\\ 1 & b & b^2\\ 1 & c & c^2\end{bmatrix} = (b-a)(c-a)(c-b).$$

19 Find the determinants of U and U^{-1} and U^2:

$$U = \begin{bmatrix}1 & 4 & 6\\ 0 & 2 & 5\\ 0 & 0 & 3\end{bmatrix} \quad\text{and}\quad U = \begin{bmatrix}a & b\\ 0 & d\end{bmatrix}.$$

20 Suppose you do two row operations at once, going from

$$\begin{bmatrix}a & b\\ c & d\end{bmatrix} \quad\text{to}\quad \begin{bmatrix}a-Lc & b-Ld\\ c-la & d-lb\end{bmatrix}.$$

Find the second determinant. Does it equal $ad - bc$?

21 *Row exchange*: Add row 1 of A to row 2, then subtract row 2 from row 1. Then add row 1 to row 2 and multiply row 1 by -1 to reach B. Which rules show

$$\det B = \begin{vmatrix}c & d\\ a & b\end{vmatrix} \quad\text{equals}\quad -\det A = -\begin{vmatrix}a & b\\ c & d\end{vmatrix}?$$

Those rules could replace Rule 2 in the definition of the determinant.

22 From $ad - bc$, find the determinants of A and A^{-1} and $A - \lambda I$:

$$A = \begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix} \quad\text{and}\quad A^{-1} = \frac{1}{3}\begin{bmatrix}2 & -1\\ -1 & 2\end{bmatrix} \quad\text{and}\quad A - \lambda I = \begin{bmatrix}2-\lambda & 1\\ 1 & 2-\lambda\end{bmatrix}.$$

Which two numbers λ lead to $\det(A - \lambda I) = 0$? Write down the matrix $A - \lambda I$ for each of those numbers λ—it should not be invertible.

23 From $A = \left[\begin{smallmatrix} 4 & 1 \\ 2 & 3 \end{smallmatrix}\right]$ find A^2 and A^{-1} and $A - \lambda I$ and their determinants. Which two numbers λ lead to $\det(A - \lambda I) = 0$?

24 Elimination reduces A to U. Then $A = LU$:

$$A = \begin{bmatrix} 3 & 3 & 4 \\ 6 & 8 & 7 \\ -3 & 5 & -9 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 4 & 1 \end{bmatrix} \begin{bmatrix} 3 & 3 & 4 \\ 0 & 2 & -1 \\ 0 & 0 & -1 \end{bmatrix} = LU.$$

Find the determinants of L, U, A, $U^{-1}L^{-1}$, and $U^{-1}L^{-1}A$.

25 If the i, j entry of A is i times j, show that $\det A = 0$. (Exception when $A = [\,1\,]$.)

26 If the i, j entry of A is $i + j$, show that $\det A = 0$. (Exception when $n = 1$ or 2.)

27 Compute the determinants of these matrices by row operations:

$$A = \begin{bmatrix} 0 & a & 0 \\ 0 & 0 & b \\ c & 0 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & a & 0 & 0 \\ 0 & 0 & b & 0 \\ 0 & 0 & 0 & c \\ d & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} a & a & a \\ a & b & b \\ a & b & c \end{bmatrix}.$$

28 True or false (give a reason if true or a 2 by 2 example if false):

(a) If A is not invertible then AB is not invertible.

(b) The determinant of A is always the product of its pivots.

(c) The determinant of $A - B$ equals $\det A - \det B$.

(d) AB and BA have the same determinant.

29 What is wrong with this proof that projection matrices have $\det P = 1$?

$$P = A(A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}} \quad \text{so} \quad |P| = |A|\frac{1}{|A^{\mathrm{T}}||A|}|A^{\mathrm{T}}| = 1.$$

30 (Calculus question) Show that the partial derivatives of $\ln(\det A)$ give A^{-1}!

$$f(a,b,c,d) = \ln(ad - bc) \quad \text{leads to} \quad \begin{bmatrix} \partial f/\partial a & \partial f/\partial c \\ \partial f/\partial b & \partial f/\partial d \end{bmatrix} = A^{-1}.$$

31 (MATLAB) The Hilbert matrix **hilb**(n) has i, j entry equal to $1/(i + j - 1)$. Print the determinants of **hilb**(1), **hilb**(2), ..., **hilb**(10). Hilbert matrices are hard to work with! What are the pivots of **hilb** (5)?

32 (MATLAB) What is a typical determinant (experimentally) of **rand**(n) and **randn**(n) for $n = 50, 100, 200, 400$? (And what does "Inf" mean in MATLAB?)

33 (MATLAB) Find the largest determinant of a 6 by 6 matrix of 1's and -1's.

34 If you know that $\det A = 6$, what is the determinant of B?

$$\text{From } \det A = \begin{vmatrix} \text{row 1} \\ \text{row 2} \\ \text{row 3} \end{vmatrix} = 6 \text{ find } \det B = \begin{vmatrix} \text{row 3 + row 2 + row 1} \\ \text{row 2 + row 1} \\ \text{row 1} \end{vmatrix}.$$

5.2 Permutations and Cofactors

A computer finds the determinant from the pivots. This section explains two other ways to do it. There is a "big formula" using all $n!$ permutations. There is a "cofactor formula" using determinants of size $n-1$. The best example is my favorite 4 by 4 matrix:

$$A = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix} \quad \text{has} \quad \det A = 5.$$

We can find this determinant in all three ways: ***pivots, big formula, cofactors.***

1. The product of the pivots is $2 \cdot \frac{3}{2} \cdot \frac{4}{3} \cdot \frac{5}{4}$. Cancellation produces 5.

2. The "big formula" in equation (8) has $4! = 24$ terms. Only five terms are nonzero:

$$\det A = 16 - 4 - 4 - 4 + 1 = 5.$$

   The 16 comes from $2 \cdot 2 \cdot 2 \cdot 2$ on the diagonal of A. Where do -4 and $+1$ come from? When you can find those five terms, you have understood formula (8).

3. The numbers $2, -1, 0, 0$ in the first row multiply their cofactors $4, 3, 2, 1$ from the other rows. That gives $2 \cdot 4 - 1 \cdot 3 = 5$. Those cofactors are 3 by 3 determinants. Cofactors use the rows and columns that are *not* used by the entry in the first row. ***Every term in a determinant uses each row and column once!***

The Pivot Formula

Elimination leaves the pivots $d_1, \ldots, d_n$ on the diagonal of the upper triangular U. If no row exchanges are involved, ***multiply those pivots*** to find the determinant:

$$\det A = (\det L)(\det U) = (1)(d_1 d_2 \cdots d_n). \tag{1}$$

This formula for $\det A$ appeared in the previous section, with the further possibility of row exchanges. The permutation matrix in $PA = LU$ has determinant -1 or $+1$. This factor $\det P = \pm 1$ enters the determinant of A:

$$(\det P)(\det A) = (\det L)(\det U) \quad \text{gives} \quad \det A = \pm(d_1 d_2 \cdots d_n). \tag{2}$$

When A has fewer than n pivots, $\det A = 0$ by Rule 8. The matrix is singular.

Example 1 A row exchange produces pivots 4, 2, 1 and that important minus sign:

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \qquad PA = \begin{bmatrix} 4 & 5 & 6 \\ 0 & 2 & 3 \\ 0 & 0 & 1 \end{bmatrix} \qquad \det A = -(4)(2)(1) = -8.$$

The odd number of row exchanges (namely one exchange) means that $\det P = -1$.

The next example has no row exchanges. It may be the first matrix we factored into LU (when it was 3 by 3). What is remarkable is that we can go directly to n by n. Pivots give the determinant. We will also see how determinants give the pivots.

Example 2 The first pivots of this tridiagonal matrix A are $2, \frac{3}{2}, \frac{4}{3}$. The next are $\frac{5}{4}$ and $\frac{6}{5}$ and eventually $\frac{n+1}{n}$. Factoring this n by n matrix reveals its determinant:

$$\begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & -1 & 2 & \cdot & \\ & & \cdot & \cdot & -1 \\ & & & -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & & & & \\ -\frac{1}{2} & 1 & & & \\ & -\frac{2}{3} & 1 & & \\ & & \cdot & \cdot & \\ & & & -\frac{n-1}{n} & 1 \end{bmatrix} \begin{bmatrix} \mathbf{2} & -1 & & & \\ & \frac{3}{2} & -1 & & \\ & & \frac{4}{3} & -1 & \\ & & & \cdot & \cdot \\ & & & & \frac{n+1}{n} \end{bmatrix}$$

The pivots are on the diagonal of U (the last matrix). When 2 and $\frac{3}{2}$ and $\frac{4}{3}$ and $\frac{5}{4}$ are multiplied, the fractions cancel. The determinant of the 4 by 4 matrix is 5. The 3 by 3 determinant is 4. *The n by n determinant is $n+1$:*

$$\textbf{−1, 2, −1 matrix} \qquad \det A = (2)\left(\tfrac{3}{2}\right)\left(\tfrac{4}{3}\right)\cdots\left(\tfrac{n+1}{n}\right) = n+1.$$

Important point: The first pivots depend only on the *upper left corner* of the original matrix A. This is a rule for all matrices without row exchanges.

> The first k pivots come from the k by k matrix A_k in the top left corner of A.
> ***The determinant of that corner submatrix A_k is $d_1 d_2 \cdots d_k$.***

The 1 by 1 matrix A_1 contains the very first pivot d_1. This is $\det A_1$. The 2 by 2 matrix in the corner has $\det A_2 = d_1 d_2$. Eventually the n by n determinant uses the product of all n pivots to give $\det A_n$ which is $\det A$.

Elimination deals with the corner matrix A_k while starting on the whole matrix. We assume no row exchanges—then $A = LU$ and $A_k = L_k U_k$. Dividing one determinant by the previous determinant ($\det A_k$ divided by $\det A_{k-1}$) cancels everything but the latest pivot d_k. ***This gives a ratio of determinants formula for the pivots:***

$$\textbf{Pivots from determinants} \qquad \textbf{The } k\textbf{th pivot is } d_k = \frac{d_1 d_2 \cdots d_k}{d_1 d_2 \cdots d_{k-1}} = \frac{\det A_k}{\det A_{k-1}}. \tag{3}$$

In the $-1, 2, -1$ matrices this ratio correctly gives the pivots $\frac{2}{1}, \frac{3}{2}, \frac{4}{3}, \ldots, \frac{n+1}{n}$. The Hilbert matrices in Problem 5.1.31 also build from the upper left corner.

We don't need row exchanges when all these corner submatrices have $\det A_k \neq 0$.

The Big Formula for Determinants

Pivots are good for computing. They concentrate a lot of information—enough to find the determinant. But it is hard to connect them to the original a_{ij}. That part will be clearer if we go back to rules 1-2-3, linearity and sign reversal and $\det I = 1$. We want to derive a single explicit formula for the determinant, directly from the entries a_{ij}.

The formula has n! terms. Its size grows fast because $n! = 1, 2, 6, 24, 120, \ldots$. For $n = 11$ there are about forty million terms. For $n = 2$, the two terms are ad and bc. Half the terms have minus signs (as in $-bc$). The other half have plus signs (as in ad). For $n = 3$ there are $3! = (3)(2)(1)$ terms. Here are those six terms:

$$\text{3 by 3 determinant} \quad \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{matrix} +a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} \\ -a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31}. \end{matrix} \tag{4}$$

Notice the pattern. Each product like $a_{11}a_{23}a_{32}$ has ***one entry from each row***. It also has ***one entry from each column***. The column order 1, 3, 2 means that this particular term comes with a minus sign. The column order 3, 1, 2 in $a_{13}a_{21}a_{32}$ has a plus sign. It will be "permutations" that tell us the sign.

The next step ($n = 4$) brings $4! = 24$ terms. There are 24 ways to choose one entry from each row and column. Down the main diagonal, $a_{11}a_{22}a_{33}a_{44}$ with column order $1, 2, 3, 4$ always has a plus sign. That is the "identity permutation".

To derive the big formula I start with $n = 2$. The goal is to reach $ad - bc$ in a systematic way. Break each row into two simpler rows:

$$\begin{bmatrix} a & b \end{bmatrix} = \begin{bmatrix} a & 0 \end{bmatrix} + \begin{bmatrix} 0 & b \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} c & d \end{bmatrix} = \begin{bmatrix} c & 0 \end{bmatrix} + \begin{bmatrix} 0 & d \end{bmatrix}.$$

Now apply linearity, first in row 1 (with row 2 fixed) and then in row 2 (with row 1 fixed):

$$\begin{aligned} \begin{vmatrix} a & b \\ c & d \end{vmatrix} &= \begin{vmatrix} a & 0 \\ c & d \end{vmatrix} + \begin{vmatrix} 0 & b \\ c & d \end{vmatrix} \\ &= \begin{vmatrix} a & 0 \\ c & 0 \end{vmatrix} + \begin{vmatrix} a & 0 \\ 0 & d \end{vmatrix} + \begin{vmatrix} 0 & b \\ c & 0 \end{vmatrix} + \begin{vmatrix} 0 & b \\ 0 & d \end{vmatrix}. \end{aligned} \tag{5}$$

The last line has $2^2 = 4$ determinants. The first and fourth are zero because their rows are dependent—one row is a multiple of the other row. We are left with $2! = 2$ determinants to compute:

$$\begin{vmatrix} a & 0 \\ 0 & d \end{vmatrix} + \begin{vmatrix} 0 & b \\ c & 0 \end{vmatrix} = ad \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} + bc \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = ad - bc.$$

The splitting led to permutation matrices. Their determinants give a plus or minus sign. The 1's are multiplied by numbers that come from A. The permutation tells the column sequence, in this case $(1, 2)$ or $(2, 1)$.

Now try $n = 3$. Each row splits into 3 simpler rows like $[a_{11} \;\; 0 \;\; 0]$. Using linearity in each row, $\det A$ splits into $3^3 = 27$ simple determinants. If a column choice is repeated—for example if we also choose $[a_{21} \;\; 0 \;\; 0]$—then the simple determinant is zero. We pay attention only when ***the nonzero terms come from different columns***.

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & & \\ & a_{22} & \\ & & a_{33} \end{vmatrix} + \begin{vmatrix} & a_{12} & \\ & & a_{23} \\ a_{31} & & \end{vmatrix} + \begin{vmatrix} & & a_{13} \\ a_{21} & & \\ & a_{32} & \end{vmatrix}$$

$$\textbf{Six terms} \quad + \begin{vmatrix} a_{11} & & \\ & & a_{23} \\ & a_{32} & \end{vmatrix} + \begin{vmatrix} & a_{12} & \\ a_{21} & & \\ & & a_{33} \end{vmatrix} + \begin{vmatrix} & & a_{13} \\ & a_{22} & \\ a_{31} & & \end{vmatrix}.$$

There are $3! = 6$ ***ways to order the columns, so six determinants.*** The six permutations of $(1, 2, 3)$ include the identity permutation $(1, 2, 3)$ from $P = I$:

$$\textbf{Column numbers} = (1,2,3), (2,3,1), (3,1,2), (1,3,2), (2,1,3), (3,2,1). \qquad (6)$$

The last three are *odd permutations* (one exchange). The first three are *even permutations* (0 or 2 exchanges). When the column sequence is (α, β, ω), we have chosen the entries $a_{1\alpha}a_{2\beta}a_{3\omega}$—and the column sequence comes with a plus or minus sign. The determinant of A is now split into six simple terms. Factor out the a_{ij}:

$$\begin{aligned} \det A = a_{11}a_{22}a_{33}\begin{vmatrix} 1 & & \\ & 1 & \\ & & 1 \end{vmatrix} + a_{12}a_{23}a_{31}\begin{vmatrix} & 1 & \\ & & 1 \\ 1 & & \end{vmatrix} + a_{13}a_{21}a_{32}\begin{vmatrix} & & 1 \\ 1 & & \\ & 1 & \end{vmatrix} \\ + a_{11}a_{23}a_{32}\begin{vmatrix} 1 & & \\ & & 1 \\ & 1 & \end{vmatrix} + a_{12}a_{21}a_{33}\begin{vmatrix} & 1 & \\ 1 & & \\ & & 1 \end{vmatrix} + a_{13}a_{22}a_{31}\begin{vmatrix} & & 1 \\ & 1 & \\ 1 & & \end{vmatrix}. \end{aligned} \qquad (7)$$

The first three (even) permutations have $\det P = +1$, the last three (odd) permutations have $\det P = -1$. We have proved the 3 by 3 formula in a systematic way.

Now you can see the n by n formula. There are $n!$ orderings of the columns. The columns $(1, 2, \ldots, n)$ go in each possible order $(\alpha, \beta, \ldots, \omega)$. Taking $a_{1\alpha}$ from row 1 and $a_{2\beta}$ from row 2 and eventually $a_{n\omega}$ from row n, the determinant contains the product $a_{1\alpha}a_{2\beta}\cdots a_{n\omega}$ times $+1$ or -1. Half the column orderings have sign -1.

The complete determinant of A is the sum of these $n!$ simple determinants, times 1 or -1. The simple determinants $a_{1\alpha}a_{2\beta}\cdots a_{n\omega}$ choose ***one entry from every row and column***:

$$\begin{aligned} \det A &= \text{sum over all } \mathbf{n!} \text{ column permutations } P = (\alpha, \beta, \ldots, \omega) \\ &= \sum (\det P) a_{1\alpha}a_{2\beta}\cdots a_{n\omega} = \textbf{BIG FORMULA}. \end{aligned} \qquad (8)$$

The 2 by 2 case is $+a_{11}a_{22} - a_{12}a_{21}$ (which is $ad - bc$). Here P is $(1, 2)$ or $(2, 1)$.

The 3 by 3 case has three products "down to the right" (see Problem 28) and three products "down to the left". Warning: Many people believe they should follow this pattern in the 4 by 4 case. They only take 8 products—but we need 24.

Example 3 (Determinant of U) When U is upper triangular, only one of the $n!$ products can be nonzero. This one term comes from the diagonal: $\det U = +u_{11}u_{22}\cdots u_{nn}$. All other column orderings pick at least one entry below the diagonal, where U has zeros. As soon as we pick a number like $u_{21} = 0$ from below the diagonal, that term in equation (8) is sure to be zero.

Of course $\det I = 1$. The only nonzero term is $+(1)(1)\cdots(1)$ from the diagonal.

Example 4 Suppose Z is the identity matrix except for column 3. Then

$$\text{determinant of } Z = \begin{vmatrix} 1 & 0 & a & 0 \\ 0 & 1 & b & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & d & 1 \end{vmatrix} = c. \tag{9}$$

The term $(1)(1)(c)(1)$ comes from the main diagonal with a plus sign. There are 23 other products (choosing one factor from each row and column) but they are all zero. Reason: If we pick a, b, or d from column 3, that column is used up. Then the only available choice from row 3 is zero.

Here is a different reason for the same answer. If $c = 0$, then Z has a row of zeros and $\det Z = c = 0$ is correct. If c is not zero, *use elimination.* Subtract multiples of row 3 from the other rows, to knock out a, b, d. That leaves a diagonal matrix and $\det Z = c$.

This example will soon be used for "Cramer's Rule". If we move a, b, c, d into the first column of Z, the determinant is $\det Z = a$. (*Why*?) Changing one column of I leaves Z with an easy determinant, coming from its main diagonal only.

Example 5 Suppose A has 1's just above and below the main diagonal. Here $n = 4$:

$$A_4 = \begin{bmatrix} 0 & \mathbf{1} & 0 & 0 \\ \mathbf{1} & 0 & 1 & 0 \\ 0 & 1 & 0 & \mathbf{1} \\ 0 & 0 & \mathbf{1} & 0 \end{bmatrix} \quad \text{and} \quad P_4 = \begin{bmatrix} 0 & \mathbf{1} & 0 & 0 \\ \mathbf{1} & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbf{1} \\ 0 & 0 & \mathbf{1} & 0 \end{bmatrix} \quad \text{have determinant 1.}$$

The only nonzero choice in the first row is column 2. The only nonzero choice in row 4 is column 3. Then rows 2 and 3 must choose columns 1 and 4. In other words P_4 is the only permutation that picks out nonzeros in A_4. The determinant of P_4 is $+1$ (two exchanges to reach $2, 1, 4, 3$). Therefore $\det A_4 = +1$.

Determinant by Cofactors

Formula (8) is a direct definition of the determinant. It gives you everything at once—but you have to digest it. Somehow this sum of $n!$ terms must satisfy rules 1-2-3 (then all the other properties follow). The easiest is $\det I = 1$, already checked. The rule of linearity becomes clear, if you ***separate out the factor a_{11} or a_{12} or $a_{1\alpha}$ that comes from the first row.*** For 3 by 3, separate the usual 6 terms of the determinant into 3 pairs:

$$\det A = a_{11}\,(a_{22}a_{33} - a_{23}a_{32}) + a_{12}\,(a_{23}a_{31} - a_{21}a_{33}) + a_{13}\,(a_{21}a_{32} - a_{22}a_{31}). \tag{10}$$

Those three quantities in parentheses are called ***"cofactors"***. They are 2 by 2 determinants, coming from matrices in rows 2 and 3. The first row contributes the factors a_{11}, a_{12}, a_{13}. The lower rows contribute the cofactors C_{11}, C_{12}, C_{13}. Certainly the determinant $a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$ depends linearly on a_{11}, a_{12}, a_{13}—this is rule 3.

The cofactor of a_{11} is $C_{11} = a_{22}a_{33} - a_{23}a_{32}$. You can see it in this splitting:

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & & \\ & a_{22} & a_{23} \\ & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} & a_{12} & \\ a_{21} & & a_{23} \\ a_{31} & & a_{33} \end{vmatrix} + \begin{vmatrix} & & a_{13} \\ a_{21} & a_{22} & \\ a_{31} & a_{32} & \end{vmatrix}.$$

We are still choosing ***one entry from each row and column***. Since a_{11} uses up row 1 and column 1, that leaves a 2 by 2 determinant as its cofactor.

As always, we have to watch signs. The 2 by 2 determinant that goes with a_{12} looks like $a_{21}a_{33} - a_{23}a_{31}$. But in the cofactor C_{12}, *its sign is reversed.* Then $a_{12}C_{12}$ is the correct 3 by 3 determinant. The sign pattern for cofactors along the first row is plus-minus-plus-minus. ***You cross out row*** 1 ***and column*** j ***to get a submatrix*** M_{1j} ***of size*** $n - 1$. Multiply its determinant by $(-1)^{1+j}$ to get the cofactor:

$$\text{The cofactors along row 1 are } C_{1j} = (-1)^{1+j} \det M_{1j}.$$

$$\textbf{The cofactor expansion is } \det A = a_{11}C_{11} + a_{12}C_{12} + \cdots + a_{1n}C_{1n}. \qquad (11)$$

In the big formula (8), the terms that multiply a_{11} combine to give $\det M_{11}$. The sign is $(-1)^{1+1}$, meaning *plus*. Equation (11) is another form of equation (8) and also equation (10), with factors from row 1 multiplying cofactors that use the other rows.

Note Whatever is possible for row 1 is possible for row i. The entries a_{ij} in that row also have cofactors C_{ij}. Those are determinants of order $n - 1$, multiplied by $(-1)^{i+j}$. Since a_{ij} accounts for row i and column j, ***the submatrix*** M_{ij} ***throws out row*** i ***and column*** j. The display shows a_{43} and M_{43} (with row 4 and column 3 removed). The sign $(-1)^{4+3}$ multiplies the determinant of M_{43} to give C_{43}. The sign matrix shows the $\pm$ pattern:

$$A = \begin{bmatrix} \bullet & \bullet & & \bullet \\ \bullet & \bullet & & \bullet \\ \bullet & \bullet & & \bullet \\ & & a_{43} & \end{bmatrix} \qquad \text{signs } (-1)^{i+j} = \begin{bmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{bmatrix}.$$

The determinant is the dot product of any row i of A with its cofactors using other rows:

$$\textbf{COFACTOR FORMULA} \qquad \det A = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}. \qquad (12)$$

Each cofactor C_{ij} (order $n - 1$, without row i and column j) includes its correct sign:

$$\textbf{Cofactor} \qquad C_{ij} = (-1)^{i+j} \det M_{ij}.$$

A determinant of order n is a combination of determinants of order $n - 1$. A recursive person would keep going. Each subdeterminant breaks into determinants of order $n - 2$. *We could define all determinants via equation* (12). This rule goes from order n to $n - 1$

to $n-2$ and eventually to order 1. Define the 1 by 1 determinant $|a|$ to be the number a. Then the cofactor method is complete.

We preferred to construct $\det A$ from its properties (linearity, sign reversal, $\det I = 1$). The big formula (8) and the cofactor formulas (10)–(12) follow from those properties. One last formula comes from the rule that $\det A = \det A^{\mathrm{T}}$. We can expand in cofactors, *down a column* instead of across a row. Down column j the entries are a_{1j} to a_{nj}. The cofactors are C_{1j} to C_{nj}. The determinant is the dot product:

$$\textbf{Cofactors down column } j: \qquad \det A = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj}. \tag{13}$$

Cofactors are useful when matrices have many zeros—as in the next examples.

Example 6 The $-1, 2, -1$ matrix has only two nonzeros in its first row. So only two cofactors C_{11} and C_{12} are involved in the determinant. I will highlight C_{12}:

$$\begin{vmatrix} 2 & -1 & & \\ \mathbf{-1} & 2 & \mathbf{-1} & \\ & -1 & \mathbf{2} & \mathbf{-1} \\ & & \mathbf{-1} & \mathbf{2} \end{vmatrix} = 2\begin{vmatrix} 2 & -1 & \\ -1 & 2 & -1 \\ & -1 & 2 \end{vmatrix} - (-1)\begin{vmatrix} \mathbf{-1} & \mathbf{-1} & \\ & \mathbf{2} & \mathbf{-1} \\ & \mathbf{-1} & \mathbf{2} \end{vmatrix}. \tag{14}$$

You see 2 times C_{11} first on the right, from crossing out row 1 and column 1. This cofactor has exactly the same $-1, 2, -1$ pattern as the original A—but one size smaller.

To compute the boldface C_{12}, *use cofactors down its first column.* The only nonzero is at the top. That contributes another -1 (so we are back to minus). Its cofactor is the $-1, 2, -1$ determinant which is 2 by 2, *two sizes smaller* than the original A.

Summary ***Each determinant*** D_n ***of order*** n ***comes from*** D_{n-1} ***and*** D_{n-2}:

$$D_4 = 2D_3 - D_2 \qquad \text{and generally} \qquad D_n = 2D_{n-1} - D_{n-2}. \tag{15}$$

Direct calculation gives $D_2 = 3$ and $D_3 = 4$. Equation (14) has $D_4 = 2(4) - 3 = 5$. These determinants 3, 4, 5 fit the formula $\boldsymbol{D_n = n+1}$. That "special tridiagonal answer" also came from the product of pivots in Example 2.

The idea behind cofactors is to reduce the order one step at a time. The determinants $D_n = n+1$ obey the recursion formula $n + 1 = 2n - (n-1)$. As they must.

Example 7 This is the same matrix, except the first entry (upper left) is now 1:

$$B_4 = \begin{bmatrix} 1 & -1 & & \\ -1 & 2 & -1 & \\ & -1 & 2 & -1 \\ & & -1 & 2 \end{bmatrix}.$$

All pivots of this matrix turn out to be 1. So its determinant is 1. How does that come from cofactors? Expanding on row 1, the cofactors all agree with Example 6. Just change $a_{11} = 2$ to $b_{11} = 1$:

$$\det B_4 = D_3 - D_2 \qquad \text{instead of} \qquad \det A_4 = 2D_3 - D_2.$$

The determinant of B_4 is $4 - 3 = 1$. The determinant of every B_n is $n - (n-1) = 1$. Problem 13 asks you to use cofactors of the *last* row. You still find $\det B_n = 1$.

■ REVIEW OF THE KEY IDEAS ■

1. With no row exchanges, $\det A$ = (*product of pivots*). In the upper left corner, $\det A_k$ = (*product of the first k pivots*).

2. Every term in the big formula (8) uses each row and column once. Half of the $n!$ terms have plus signs (when $\det P = +1$) and half have minus signs.

3. The cofactor C_{ij} is $(-1)^{i+j}$ times the smaller determinant that omits row i and column j (because a_{ij} uses that row and column).

4. The determinant is the dot product of any row of A with its row of cofactors. When a row of A has a lot of zeros, we only need a few cofactors.

■ WORKED EXAMPLES ■

5.2 A A *Hessenberg matrix* is a triangular matrix with one extra diagonal. Use cofactors of row 1 to show that the 4 by 4 determinant satisfies Fibonacci's rule $|H_4| = |H_3| + |H_2|$. The same rule will continue for all sizes, $|H_n| = |H_{n-1}| + |H_{n-2}|$. Which Fibonacci number is $|H_n|$?

$$H_2 = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \qquad H_3 = \begin{bmatrix} 2 & 1 & \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \qquad H_4 = \begin{bmatrix} 2 & 1 & & \\ \mathbf{1} & 2 & \mathbf{1} & \\ \mathbf{1} & 1 & \mathbf{2} & \mathbf{1} \\ \mathbf{1} & 1 & \mathbf{1} & \mathbf{2} \end{bmatrix}$$

Solution The cofactor C_{11} for H_4 is the determinant $|H_3|$. We also need C_{12} (in boldface):

$$C_{12} = -\begin{vmatrix} \mathbf{1} & \mathbf{1} & \mathbf{0} \\ \mathbf{1} & \mathbf{2} & \mathbf{1} \\ \mathbf{1} & \mathbf{1} & \mathbf{2} \end{vmatrix} = -\begin{vmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{vmatrix} + \begin{vmatrix} 1 & 0 & 0 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{vmatrix}$$

Rows 2 and 3 stayed the same and we used linearity in row 1. The two determinants on the right are $-|H_3|$ and $+|H_2|$. Then the 4 by 4 determinant is

$$|H_4| = 2C_{11} + 1C_{12} = 2|H_3| - |H_3| + |H_2| = |H_3| + |H_2|.$$

The actual numbers are $|H_2| = 3$ and $|H_3| = 5$ (and of course $|H_1| = 2$). Since $|H_n|$ follows Fibonacci's rule $|H_{n-1}| + |H_{n-2}|$, it must be $|H_n| = F_{n+2}$.

5.2 B These questions use the $\pm$ signs (even and odd P's) in the big formula for $\det A$:

1. If A is the 10 by 10 all-ones matrix, how does the big formula give $\det A = 0$?

2. If you multiply all $n!$ permutations together into a single P, is P odd or even?

3. If you multiply each a_{ij} by the fraction i/j, why is $\det A$ unchanged?

Solution In Question **1**, with all $a_{ij} = 1$, all the products in the big formula (8) will be 1. Half of them come with a plus sign, and half with minus. So they cancel to leave $\det A = 0$. (Of course the all-ones matrix is singular.)

In Question **2**, multiplying $\left[\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right]\left[\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right]$ gives an odd permutation. Also for 3 by 3, the three odd permutations multiply (in any order) to give *odd*. But for $n > 3$ the product of all permutations will be *even*. There are $n!/2$ odd permutations and that is an even number as soon as it includes the factor 4.

In Question **3**, each a_{ij} is multiplied by i/j. So each product $a_{1\alpha}a_{2\beta}\cdots a_{n\omega}$ in the big formula is multiplied by all the row numbers $i = 1, 2, \ldots, n$ and divided by all the column numbers $j = 1, 2, \ldots, n$. (The columns come in some permuted order!) Then each product is unchanged and $\det A$ stays the same.

Another approach to Question **3**: We are multiplying the matrix A by the diagonal matrix $D = \mathbf{diag}(1 : n)$ when row i is multiplied by i. And we are postmultiplying by D^{-1} when column j is divided by j. The determinant of DAD^{-1} is the same as $\det A$ by the product rule.

Problem Set 5.2

Problems 1–10 use the big formula with $n!$ terms: $|A| = \sum \pm a_{1\alpha}a_{2\beta}\cdots a_{n\omega}$.

1 Compute the determinants of A, B, C from six terms. Are their rows independent?

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 3 & 2 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 4 & 4 \\ 5 & 6 & 7 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

2 Compute the determinants of A, B, C, D. Are their columns independent?

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \quad C = \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix} \quad D = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}.$$

3 Show that $\det A = 0$, regardless of the five nonzeros marked by x's:

$$A = \begin{bmatrix} x & x & x \\ 0 & 0 & x \\ 0 & 0 & x \end{bmatrix}.$$

What are the cofactors of row 1?
What is the rank of A?
What are the 6 terms in $\det A$?

4 Find two ways to choose nonzeros from four different rows and columns:

$$A = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 3 & 4 & 5 \\ 5 & 4 & 0 & 3 \\ 2 & 0 & 0 & 1 \end{bmatrix} \quad (B \text{ has the same zeros as } A).$$

Is $\det A$ equal to $1 + 1$ or $1 - 1$ or $-1 - 1$? What is $\det B$?

5 Place the smallest number of zeros in a 4 by 4 matrix that will guarantee $\det A = 0$. Place as many zeros as possible while still allowing $\det A \neq 0$.

6 (a) If $a_{11} = a_{22} = a_{33} = 0$, how many of the six terms in $\det A$ will be zero?

(b) If $a_{11} = a_{22} = a_{33} = a_{44} = 0$, how many of the 24 products $a_{1j}a_{2k}a_{3l}a_{4m}$ are sure to be zero?

7 How many 5 by 5 permutation matrices have $\det P = +1$? Those are even permutations. Find one that needs four exchanges to reach the identity matrix.

8 If $\det A$ is not zero, at least one of the $n!$ terms in formula (8) is not zero. Deduce from the big formula that some ordering of the rows of A leaves no zeros on the diagonal. (Don't use P from elimination; that PA can have zeros on the diagonal.)

9 Show that 4 is the largest determinant for a 3 by 3 matrix of 1's and -1's.

10 How many permutations of $(1, 2, 3, 4)$ are even and what are they? Extra credit: What are all the possible 4 by 4 determinants of $I + P_{\text{even}}$?

Problems 11–22 use cofactors $C_{ij} = (-1)^{i+j} \det M_{ij}$. Remove row i and column j.

11 Find all cofactors and put them into cofactor matrices C, D. Find AC and $\det B$.

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 0 & 0 \end{bmatrix}.$$

12 Find the cofactor matrix C and multiply A times C^{T}. Compare AC^{T} with A^{-1}:

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \qquad A^{-1} = \frac{1}{4}\begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{bmatrix}.$$

13 The n by n determinant C_n has 1's above and below the main diagonal:

$$C_1 = |0| \quad C_2 = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} \quad C_3 = \begin{vmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{vmatrix} \quad C_4 = \begin{vmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{vmatrix}.$$

(a) What are these determinants C_1, C_2, C_3, C_4?

(b) By cofactors find the relation between C_n and C_{n-1} and C_{n-2}. Find C_{10}.

14 The matrices in Problem 13 have 1's just above and below the main diagonal. Going down the matrix, which order of columns (if any) gives all 1's? Explain why that permutation is *even* for $n = 4, 8, 12, \ldots$ and *odd* for $n = 2, 6, 10, \ldots$. Then

$$C_n = 0 \text{ (odd } n) \qquad C_n = 1 \; (n = 4, 8, \cdots) \qquad C_n = -1 \; (n = 2, 6, \cdots).$$

15 The tridiagonal 1, 1, 1 matrix of order n has determinant E_n:

$$E_1 = |1| \quad E_2 = \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} \quad E_3 = \begin{vmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{vmatrix} \quad E_4 = \begin{vmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{vmatrix}.$$

(a) By cofactors show that $E_n = E_{n-1} - E_{n-2}$.

(b) Starting from $E_1 = 1$ and $E_2 = 0$ find $E_3, E_4, \ldots, E_8$.

(c) By noticing how these numbers eventually repeat, find E_{100}.

16 F_n is the determinant of the $1, 1, -1$ tridiagonal matrix of order n:

$$F_2 = \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} = 2 \quad F_3 = \begin{vmatrix} 1 & -1 & 0 \\ 1 & 1 & -1 \\ 0 & 1 & 1 \end{vmatrix} = 3 \quad F_4 = \begin{vmatrix} 1 & -1 & & \\ 1 & 1 & -1 & \\ & 1 & 1 & -1 \\ & & 1 & 1 \end{vmatrix} \neq 4.$$

Expand in cofactors to show that $F_n = F_{n-1} + F_{n-2}$. These determinants are *Fibonacci numbers* $1, 2, 3, 5, 8, 13, \ldots$. The sequence usually starts $1, 1, 2, 3$ (with two 1's) so our F_n is the usual F_{n+1}.

17 The matrix B_n is the $-1, 2, -1$ matrix A_n except that $b_{11} = 1$ instead of $a_{11} = 2$. Using cofactors of the *last* row of B_4 show that $|B_4| = 2|B_3| - |B_2| = 1$.

$$B_4 = \begin{bmatrix} \mathbf{1} & -1 & & \\ -1 & 2 & -1 & \\ & -1 & 2 & -1 \\ & & -1 & 2 \end{bmatrix} \quad B_3 = \begin{bmatrix} \mathbf{1} & -1 & \\ -1 & 2 & -1 \\ & -1 & 2 \end{bmatrix} \quad B_2 = \begin{bmatrix} \mathbf{1} & -1 \\ -1 & 2 \end{bmatrix}.$$

The recursion $|B_n| = 2|B_{n-1}| - |B_{n-2}|$ is satisfied when every $|B_n| = 1$. This recursion is the same as for the A's in Example 6. The difference is in the starting values 1, 1, 1 for the determinants of sizes $n = 1, 2, 3$.

18 Go back to B_n in Problem 17. It is the same as A_n except for $b_{11} = 1$. So use linearity in the first row, where $[1 \ \ -1 \ \ 0]$ equals $[2 \ \ -1 \ \ 0]$ minus $[1 \ \ 0 \ \ 0]$:

$$|B_n| = \begin{vmatrix} 1 & -1 & & 0 \\ -1 & & & \\ & & A_{n-1} & \\ 0 & & & \end{vmatrix} = \begin{vmatrix} 2 & -1 & & 0 \\ -1 & & & \\ & & A_{n-1} & \\ 0 & & & \end{vmatrix} - \begin{vmatrix} 1 & 0 & & 0 \\ -1 & & & \\ & & A_{n-1} & \\ 0 & & & \end{vmatrix}.$$

Linearity gives $|B_n| = |A_n| - |A_{n-1}| =$ _____.

19 Explain why the 4 by 4 Vandermonde determinant contains x^3 but not x^4 or x^5:

$$V_4 = \det \begin{bmatrix} 1 & a & a^2 & a^3 \\ 1 & b & b^2 & b^3 \\ 1 & c & c^2 & c^3 \\ 1 & x & x^2 & x^3 \end{bmatrix}.$$

The determinant is zero at $x =$ _____, _____, and _____. The cofactor of x^3 is $V_3 = (b-a)(c-a)(c-b)$. Then $V_4 = (b-a)(c-a)(c-b)(x-a)(x-b)(x-c)$.

20 Find G_2 and G_3 and then by row operations G_4. Can you predict G_n?

$$G_2 = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} \qquad G_3 = \begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix} \qquad G_4 = \begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{vmatrix}.$$

21 Compute S_1, S_2, S_3 for these 1, 3, 1 matrices. By Fibonacci guess and check S_4.

$$S_1 = |3| \qquad S_2 = \begin{vmatrix} 3 & 1 \\ 1 & 3 \end{vmatrix} \qquad S_3 = \begin{vmatrix} 3 & 1 & 0 \\ 1 & 3 & 1 \\ 0 & 1 & 3 \end{vmatrix}$$

22 Change 3 to 2 in the upper left corner of the matrices in Problem 21. Why does that subtract S_{n-1} from the determinant S_n? Show that the determinants of the new matrices become the Fibonacci numbers 2, 5, 13 (always F_{2n+1}).

Problems 23–26 are about block matrices and block determinants.

23 With 2 by 2 blocks in 4 by 4 matrices, you cannot always use block determinants:

$$\begin{vmatrix} A & B \\ 0 & D \end{vmatrix} = |A|\,|D| \qquad \text{but} \qquad \begin{vmatrix} A & B \\ C & D \end{vmatrix} \neq |A|\,|D| - |C|\,|B|.$$

(a) Why is the first statement true? Somehow B doesn't enter.

(b) Show by example that equality fails (as shown) when C enters.

(c) Show by example that the answer $\det(AD - CB)$ is also wrong.

24 With block multiplication, $A = LU$ has $A_k = L_k U_k$ in the top left corner:

$$A = \begin{bmatrix} A_k & * \\ * & * \end{bmatrix} = \begin{bmatrix} L_k & 0 \\ * & * \end{bmatrix} \begin{bmatrix} U_k & * \\ 0 & * \end{bmatrix}.$$

(a) Suppose the first three pivots of A are $2, 3, -1$. What are the determinants of L_1, L_2, L_3 (with diagonal 1's) and U_1, U_2, U_3 and A_1, A_2, A_3?

(b) If A_1, A_2, A_3 have determinants $5, 6, 7$ find the three pivots from equation (3).

25 Block elimination subtracts CA^{-1} times the first row $[\,A \quad B\,]$ from the second row $[C \quad D]$. This leaves the *Schur complement* $D - CA^{-1}B$ in the corner:

$$\begin{bmatrix} I & 0 \\ -CA^{-1} & I \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} A & B \\ 0 & D - CA^{-1}B \end{bmatrix}.$$

Take determinants of these block matrices to prove correct rules if A^{-1} exists:

$$\begin{vmatrix} A & B \\ C & D \end{vmatrix} = |A|\,|D - CA^{-1}B| = |AD - CB| \text{ provided } AC = CA.$$

26 If A is m by n and B is n by m, block multiplication gives $\det M = \det AB$:

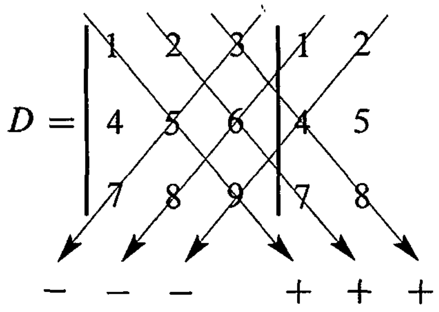
$$M = \begin{bmatrix} 0 & A \\ -B & I \end{bmatrix} = \begin{bmatrix} AB & A \\ 0 & I \end{bmatrix} \begin{bmatrix} I & 0 \\ -B & I \end{bmatrix}.$$

If A is a single row and B is a single column what is $\det M$? If A is a column and B is a row what is $\det M$? Do a 3 by 3 example of each.

27 (A calculus question) Show that the derivative of $\det A$ with respect to a_{11} is the cofactor C_{11}. The other entries are fixed—we are only changing a_{11}.

Problems 28–33 are about the "big formula" with $n!$ terms.

28 A 3 by 3 determinant has three products "down to the right" and three "down to the left" with minus signs. Compute the six terms like $(1)(5)(9) = 45$ to find D.



Explain without determinants why this particular matrix is or is not invertible.

29 For E_4 in Problem 15, five of the $4! = 24$ terms in the big formula (8) are nonzero. Find those five terms to show that $E_4 = -1$.

30 For the 4 by 4 tridiagonal second difference matrix (entries -1, 2, -1) find the five terms in the big formula that give $\det A = 16 - 4 - 4 - 4 + 1$.

31 Find the determinant of this cyclic P by cofactors of row 1 and then the "big formula". How many exchanges reorder 4, 1, 2, 3 into 1, 2, 3, 4? Is $|P^2| = 1$ or -1?

$$P = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \qquad P^2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}.$$

Challenge Problems

32 Cofactors of the 1, 3, 1 matrices in Problem 21 give a recursion $S_n = 3S_{n-1} - S_{n-2}$. Amazingly that recursion produces every second Fibonacci number. Here is the challenge.

Show that S_n is the Fibonacci number F_{2n+2} by proving $F_{2n+2} = 3F_{2n} - F_{2n-2}$. Keep using Fibonacci's rule $F_k = F_{k-1} + F_{k-2}$ starting with $k = 2n + 2$.

33 The symmetric Pascal matrices have determinant 1. If I subtract 1 from the n, n entry, why does the determinant become zero? (Use rule 3 or cofactors.)

$$\det \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & 20 \end{bmatrix} = 1 \text{ (known)} \qquad \det \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & \mathbf{19} \end{bmatrix} = \mathbf{0} \text{ (to explain)}.$$

34 This problem shows in two ways that $\det A = 0$ (the x's are any numbers):

$$A = \begin{bmatrix} x & x & x & x & x \\ x & x & x & x & x \\ 0 & 0 & 0 & x & x \\ 0 & 0 & 0 & x & x \\ 0 & 0 & 0 & x & x \end{bmatrix}.$$

(a) How do you know that the rows are linearly dependent?

(b) Explain why all 120 terms are zero in the big formula for $\det A$.

35 If $|\det(A)| > 1$, prove that the powers A^n cannot stay bounded. But if $|\det(A)| \leq 1$, show that some entries of A^n might still grow large. Eigenvalues will give the right test for stability, determinants tell us only one number.

5.3 Cramer's Rule, Inverses, and Volumes

This section solves $Ax = b$—by algebra and not by elimination. We also invert A. In the entries of A^{-1}, you will see $\det A$ in every denominator—we divide by it. (If $\det A = 0$ then we can't divide and A^{-1} doesn't exist.) Each entry in A^{-1} and $A^{-1}b$ is a determinant divided by the determinant of A.

Cramer's Rule solves $Ax = b$. A neat idea gives the first component x_1. Replacing the first column of I by x gives a matrix with determinant x_1. When you multiply it by A, the first column becomes Ax which is b. The other columns are copied from A:

Key idea $$\begin{bmatrix} & & \\ & A & \\ & & \end{bmatrix}\begin{bmatrix} x_1 & 0 & 0 \\ x_2 & 1 & 0 \\ x_3 & 0 & 1 \end{bmatrix} = \begin{bmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{bmatrix} = B_1. \qquad (1)$$

We multiplied a column at a time. *Take determinants of the three matrices*:

Product rule $$(\det A)(x_1) = \det B_1 \qquad \text{or} \qquad x_1 = \frac{\det B_1}{\det A}. \qquad (2)$$

This is the first component of x in Cramer's Rule! Changing a column of A gives B_1.

To find x_2, put the vector x into the *second* column of the identity matrix:

Same idea $$\begin{bmatrix} & & \\ a_1 & a_2 & a_3 \\ & & \end{bmatrix}\begin{bmatrix} 1 & x_1 & 0 \\ 0 & x_2 & 0 \\ 0 & x_3 & 1 \end{bmatrix} = \begin{bmatrix} & & \\ a_1 & b & a_3 \\ & & \end{bmatrix} = B_2. \qquad (3)$$

Take determinants to find $(\det A)(x_2) = \det B_2$. This gives x_2 in Cramer's Rule:

CRAMER's RULE If $\det A$ is not zero, $Ax = b$ is solved by determinants:

$$x_1 = \frac{\det B_1}{\det A} \qquad x_2 = \frac{\det B_2}{\det A} \qquad \dots \qquad x_n = \frac{\det B_n}{\det A} \qquad (4)$$

The matrix B_j has the jth column of A replaced by the vector b.

Example 1 Solving $3x_1 + 4x_2 = \mathbf{2}$ and $5x_1 + 6x_2 = \mathbf{4}$ needs three determinants:

$$\det A = \begin{vmatrix} 3 & 4 \\ 5 & 6 \end{vmatrix} \qquad \det B_1 = \begin{vmatrix} \mathbf{2} & 4 \\ \mathbf{4} & 6 \end{vmatrix} \qquad \det B_2 = \begin{vmatrix} 3 & \mathbf{2} \\ 5 & \mathbf{4} \end{vmatrix}$$

Those determinants are -2 and -4 and 2. All ratios divide by $\det A$:

Cramer's Rule $$x_1 = \frac{-4}{-2} = 2 \quad x_2 = \frac{2}{-2} = -1 \quad \textbf{check} \begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}.$$

To solve an n by n system, Cramer's Rule evaluates $n + 1$ determinants (of A and the n different B's). When each one is the sum of $n!$ terms—applying the "big formula" with all permutations—this makes a total of $(n + 1)!$ terms. *It would be crazy to solve equations that way.* But we do finally have an explicit formula for the solution x.

Example 2 Cramer's Rule is inefficient for numbers but it is well suited to letters. For $n = 2$, find the columns of A^{-1} by solving $AA^{-1} = I$:

$$\textbf{Columns of } I \qquad \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Those share the same A. We need five determinants for x_1, x_2, y_1, y_2:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} \text{ and } \begin{vmatrix} \mathbf{1} & b \\ \mathbf{0} & d \end{vmatrix} \quad \begin{vmatrix} a & \mathbf{1} \\ c & \mathbf{0} \end{vmatrix} \quad \begin{vmatrix} \mathbf{0} & b \\ \mathbf{1} & d \end{vmatrix} \quad \begin{vmatrix} a & \mathbf{0} \\ c & \mathbf{1} \end{vmatrix}$$

The last four are $d, -c, -b$, and a. (They are the cofactors!) Here is A^{-1}:

$$x_1 = \frac{d}{|A|},\ x_2 = \frac{-c}{|A|},\ y_1 = \frac{-b}{|A|},\ y_2 = \frac{a}{|A|}, \text{ and then } A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

I chose 2 by 2 so that the main points could come through clearly. The new idea is the appearance of the cofactors. When the right side is a column of the identity matrix I, the determinant of each matrix B_j in Cramer's Rule is a cofactor.

You can see those cofactors for $n = 3$. Solve $AA^{-1} = I$ (first column only):

$$\begin{matrix}\textbf{Determinants} \\ \textbf{= Cofactors of } A\end{matrix} \qquad \begin{vmatrix} \mathbf{1} & a_{12} & a_{13} \\ \mathbf{0} & a_{22} & a_{23} \\ \mathbf{0} & a_{32} & a_{33} \end{vmatrix} \quad \begin{vmatrix} a_{11} & \mathbf{1} & a_{13} \\ a_{21} & \mathbf{0} & a_{23} \\ a_{31} & \mathbf{0} & a_{33} \end{vmatrix} \quad \begin{vmatrix} a_{11} & a_{12} & \mathbf{1} \\ a_{21} & a_{22} & \mathbf{0} \\ a_{31} & a_{32} & \mathbf{0} \end{vmatrix} \qquad (5)$$

That first determinant $|B_1|$ is the cofactor C_{11}. The second determinant $|B_2|$ is the cofactor C_{12}. *Notice that* the correct minus sign appears in $-(a_{21}a_{33} - a_{23}a_{31})$. This cofactor C_{12} goes into the $2, 1$ entry of A^{-1}—the first column! So we transpose the cofactor matrix, and as always we divide by $\det A$.

The i, j entry of A^{-1} is the cofactor C_{ji} (not C_{ij}) divided by $\det A$:

$$\textbf{FORMULA FOR } A^{-1} \qquad (A^{-1})_{ij} = \frac{C_{ji}}{\det A} \quad \text{and} \quad A^{-1} = \frac{C^{\mathrm{T}}}{\det A}. \qquad (6)$$

The cofactors C_{ij} go into the *"cofactor matrix"* C. ***Its transpose leads to*** A^{-1}. To compute the i, j entry of A^{-1}, cross out row j and column i of A. Multiply the determinant by $(-1)^{i+j}$ to get the cofactor, and divide by $\det A$.

Check this rule for the $3, 1$ entry of A^{-1}. This is in column 1 so we solve $Ax = (1, 0, 0)$. The third component x_3 needs the third determinant in equation (5), divided by $\det A$. That third determinant is exactly the cofactor $C_{13} = a_{21}a_{32} - a_{22}a_{31}$. So $(A^{-1})_{31} = C_{13}/\det A$ (2 by 2 determinant divided by 3 by 3).

Summary In solving $AA^{-1} = I$, the columns of I lead to the columns of A^{-1}. Then Cramer's Rule using $\boldsymbol{b} =$ columns of I gives the short formula (6) for A^{-1}.

Direct proof of the formula $A^{-1} = C^{\mathrm{T}}/\det A$ The idea is to multiply A times C^{T}:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix} = \begin{bmatrix} \det A & 0 & 0 \\ 0 & \det A & 0 \\ 0 & 0 & \det A \end{bmatrix}. \qquad (7)$$

Row 1 of A times column 1 of the cofactors yields the first $\det A$ on the right:

$$a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} = \det A \quad \text{by the cofactor rule.}$$

Similarly row 2 of A times column 2 of C^{T}(*transpose*) yields $\det A$. The entries a_{2j} are multiplying cofactors C_{2j} as they should, to give the determinant.

How to explain the zeros off the main diagonal in equation (7)? Rows of A are multiplying cofactors from *different* rows. Why is the answer zero?

Row 2 of A
Row 1 of C

$$a_{21}C_{11} + a_{22}C_{12} + a_{23}C_{13} = 0. \qquad (8)$$

Answer: This is the cofactor rule for a new matrix, when the second row of A is copied into its first row. The new matrix A^* has two equal rows, so $\det A^* = 0$ in equation (8). Notice that A^* has the same cofactors C_{11}, C_{12}, C_{13} as A—because all rows agree after the first row. Thus the remarkable multiplication (7) is correct:

$$AC^{\mathrm{T}} = (\det A)I \qquad \text{or} \qquad A^{-1} = \frac{C^{\mathrm{T}}}{\det A}.$$

Example 3 The "sum matrix" A has determinant 1. Then A^{-1} contains cofactors:

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} \quad \text{has inverse} \quad A^{-1} = \frac{C^{\mathrm{T}}}{1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}.$$

Cross out row 1 and column 1 of A to see the 3 by 3 cofactor $C_{11} = 1$. Now cross out row 1 and column 2 for C_{12}. The 3 by 3 submatrix is still triangular with determinant 1. But the cofactor C_{12} is -1 because of the sign $(-1)^{1+2}$. This number -1 goes into the $(2,1)$ entry of A^{-1}—don't forget to transpose C.

The inverse of a triangular matrix is triangular. Cofactors give a reason why.

Example 4 If all cofactors are nonzero, is A sure to be invertible? *No way.*

Area of a Triangle

Everybody knows the area of a rectangle—base times height. The area of a triangle is *half* the base times the height. But here is a question that those formulas don't answer. ***If we know the corners*** (x_1, y_1) ***and*** (x_2, y_2) ***and*** (x_3, y_3) ***of a triangle, what is the area?*** Using the corners to find the base and height is not a good way.

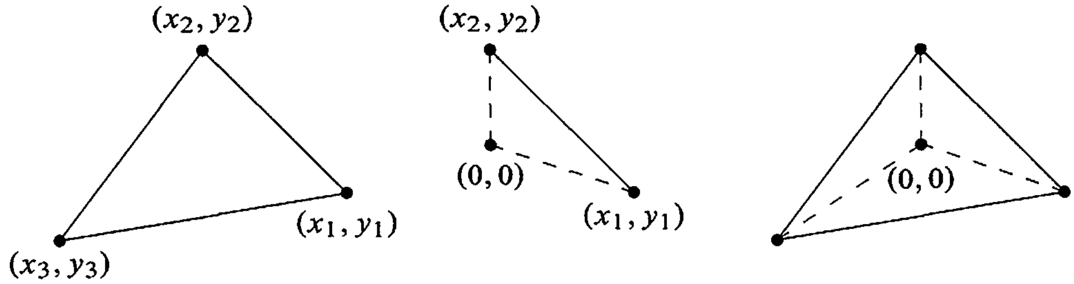


Figure 5.1: General triangle; special triangle from $(0,0)$; general from three specials.

Determinants are much better. The square roots in the base and height cancel out in the good formula. ***The area of a triangle is half of a* 3 *by* 3 *determinant.*** If one corner is at the origin, say $(x_3, y_3) = (0,0)$, the determinant is only 2 by 2.

The triangle with corners (x_1, y_1) and (x_2, y_2) and (x_3, y_3) has area $= \dfrac{\text{determinant}}{2}$:

Area of triangle $\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$ Area $= \frac{1}{2}\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix}$ when $(x_3, y_3) = (0,0)$.

When you set $x_3 = y_3 = 0$ in the 3 by 3 determinant, you get the 2 by 2 determinant. These formulas have no square roots—they are reasonable to memorize. The 3 by 3 determinant breaks into a sum of three 2 by 2's, just as the third triangle in Figure 5.1 breaks into three special triangles from $(0,0)$:

Cofactors of column 3

$$\textbf{Area} = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \begin{matrix} +\frac{1}{2}(x_1y_2 - x_2y_1) \\ +\frac{1}{2}(x_2y_3 - x_3y_2) \\ +\frac{1}{2}(x_3y_1 - x_1y_3). \end{matrix} \qquad (9)$$

If $(0,0)$ is outside the triangle, two of the special areas can be negative—but the sum is still correct. The real problem is to explain the special area $\frac{1}{2}(x_1y_2 - x_2y_1)$.

Why is this the area of a triangle? We can remove the factor $\frac{1}{2}$ and change to a parallelogram (twice as big, because the parallelogram contains two equal triangles). We now prove that the parallelogram area is the determinant $x_1y_2 - x_2y_1$. This area in Figure 5.2 is 11, and therefore the triangle has area $\frac{11}{2}$.

Proof that a parallelogram starting from* $(0,0)$ *has area* $=$ 2 *by* 2 *determinant.

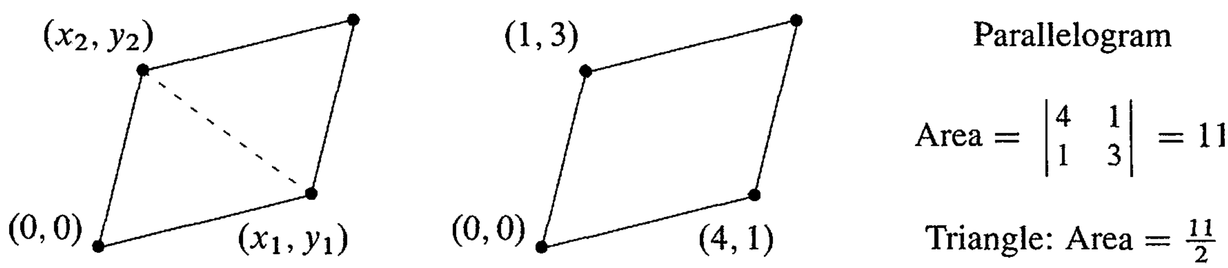


Figure 5.2: A triangle is half of a parallelogram. Area is half of a determinant.

There are many proofs but this one fits with the book. We show that the area has the same properties 1-2-3 as the determinant. Then area = determinant! Remember that those three rules defined the determinant and led to all its other properties.

1 When $A = I$, the parallelogram becomes the unit square. Its area is $\det I = 1$.

2 When rows are exchanged, the determinant reverses sign. The absolute value (positive area) stays the same—it is the same parallelogram.

3 If row 1 is multiplied by t, Figure 5.3a shows that the area is also multiplied by t. Suppose a new row (x_1', y_1') is added to (x_1, y_1) (keeping row 2 fixed). Figure 5.3b shows that the solid parallelogram areas add to the dotted parallelogram area (because the two triangles completed by dotted lines are the same).

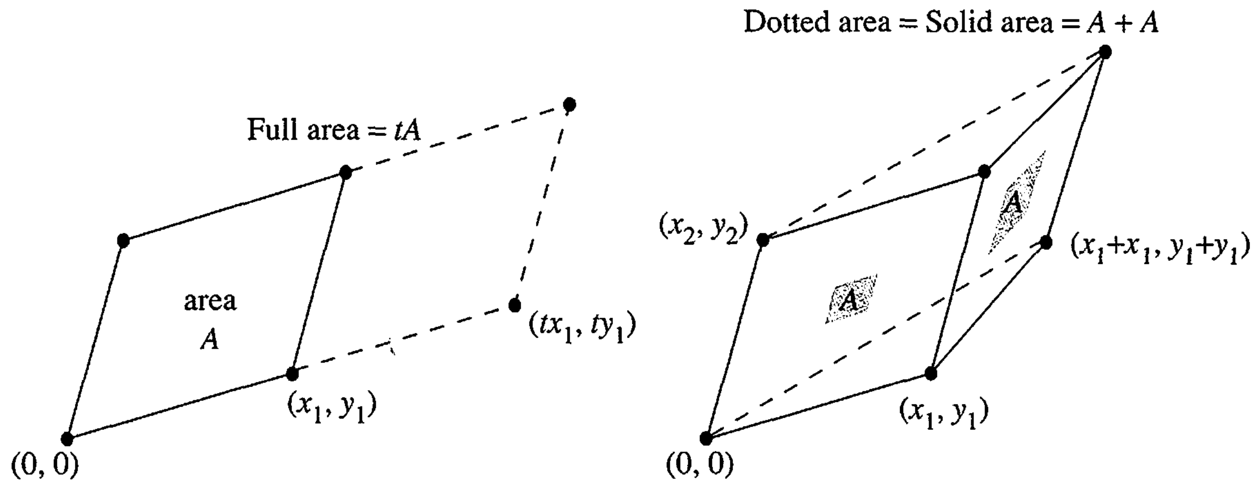


Figure 5.3: Areas obey the rule of linearity (keeping the side (x_2, y_2) constant).

That is an exotic proof, when we could use plane geometry. But the proof has a major attraction—it applies in n dimensions. The n edges going out from the origin are given by the *rows of an n by n matrix*. The box is completed by more edges, just like the parallelogram.

Figure 5.4 shows a three-dimensional box—whose edges are not at right angles. ***The volume equals the absolute value of*** $\det A$. Our proof checks again that rules 1–3 for determinants are also obeyed by volumes. When an edge is stretched by a factor t, the volume is multiplied by t. When edge 1 is added to edge $1'$, the new box has edge $1 + 1'$. Its volume is the sum of the two original volumes. This is Figure 5.3b lifted into three dimensions or n dimensions. I would draw the boxes but this paper is only two-dimensional.

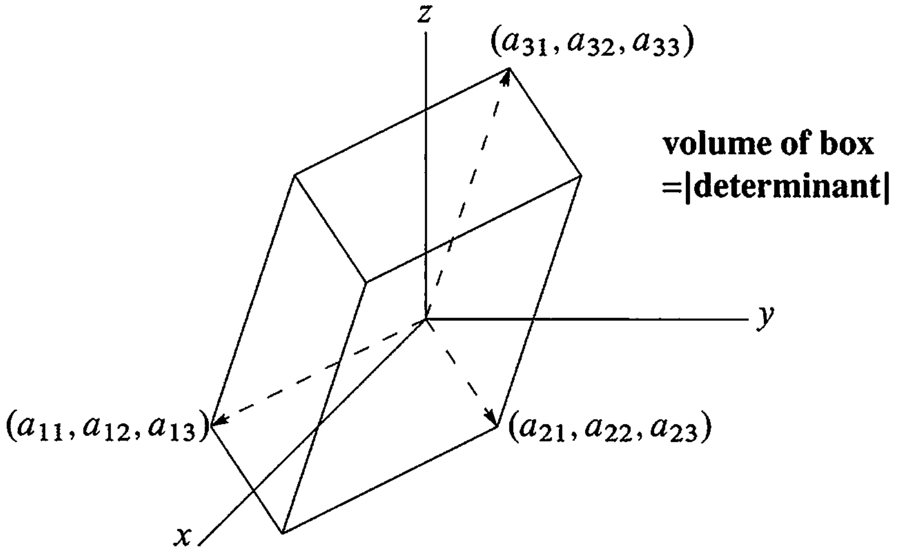


Figure 5.4: Three-dimensional box formed from the three rows of A.

The unit cube has volume = 1, which is $\det I$. Row exchanges or edge exchanges leave the same box and the same absolute volume. The determinant changes sign, to indicate whether the edges are a *right-handed triple* ($\det A > 0$) or a *left-handed triple* ($\det A < 0$). The box volume follows the rules for determinants, so volume of the box = absolute value of the determinant.

Example 5 Suppose a rectangular box (90° angles) has side lengths r, s, and t. Its volume is r times s times t. The diagonal matrix with entries r, s, and t produces those three sides. Then $\det A$ also equals $r\,s\,t$.

Example 6 In calculus, the box is infinitesimally small! To integrate over a circle, we might change x and y to r and θ. Those are polar coordinates: $x = r\cos\theta$ and $y = r\sin\theta$. The area of a "polar box" is a determinant J times $dr\,d\theta$:

$$J = \begin{vmatrix} \partial x/\partial r & \partial x/\partial\theta \\ \partial y/\partial r & \partial y/\partial\theta \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r.$$

This determinant is the r in the small area $dA = r\,dr\,d\theta$. The stretching factor J goes into double integrals just as dx/du goes into an ordinary integral $\int dx = \int (dx/du)\,du$. For triple integrals the Jacobian matrix J with nine derivatives will be 3 by 3.

The Cross Product

The *cross product* is an extra (and optional) application, special for three dimensions. Start with vectors $\boldsymbol{u} = (u_1, u_2, u_3)$ and $\boldsymbol{v} = (v_1, v_2, v_3)$. Unlike the dot product, which is a number, the cross product is a vector—also in three dimensions. It is written $\boldsymbol{u} \times \boldsymbol{v}$ and pronounced "$\boldsymbol{u}$ cross $\boldsymbol{v}$." *The components of this cross product are just* 2 *by* 2 *cofactors.* We will explain the properties that make $\boldsymbol{u} \times \boldsymbol{v}$ useful in geometry and physics.

This time we bite the bullet, and write down the formula before the properties.

DEFINITION The *cross product* of $\boldsymbol{u} = (u_1, u_2, u_3)$ and $\boldsymbol{v} = (v_1, v_2, v_3)$ is a vector

$$\boldsymbol{u} \times \boldsymbol{v} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = (u_2v_3 - u_3v_2)\boldsymbol{i} + (u_3v_1 - u_1v_3)\boldsymbol{j} + (u_1v_2 - u_2v_1)\boldsymbol{k}. \quad (10)$$

This vector is perpendicular to u and v. The cross product $\boldsymbol{v} \times \boldsymbol{u}$ is $-(\boldsymbol{u} \times \boldsymbol{v})$.

Comment The 3 by 3 determinant is the easiest way to remember $\boldsymbol{u} \times \boldsymbol{v}$. It is not especially legal, because the first row contains vectors $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ and the other rows contain numbers. In the determinant, the vector $\boldsymbol{i} = (1, 0, 0)$ multiplies u_2v_3 and $-u_3v_2$. The result is $(u_2v_3 - u_3v_2, 0, 0)$, which displays the first component of the cross product.

Notice the cyclic pattern of the subscripts: 2 and 3 give component 1 of $\boldsymbol{u} \times \boldsymbol{v}$, then 3 and 1 give component 2, then 1 and 2 give component 3. This completes the definition of $\boldsymbol{u} \times \boldsymbol{v}$. Now we list the properties of the cross product:

Property 1 $\boldsymbol{v} \times \boldsymbol{u}$ reverses rows 2 and 3 in the determinant so it equals $-(\boldsymbol{u} \times \boldsymbol{v})$.

Property 2 The cross product $\boldsymbol{u} \times \boldsymbol{v}$ is perpendicular to $\boldsymbol{u}$ (and also to $\boldsymbol{v}$). The direct proof is to watch terms cancel. Perpendicularity is a zero dot product:

$$\boldsymbol{u} \cdot (\boldsymbol{u} \times \boldsymbol{v}) = u_1(u_2v_3 - u_3v_2) + u_2(u_3v_1 - u_1v_3) + u_3(u_1v_2 - u_2v_1) = 0. \quad (11)$$

The determinant now has rows $\boldsymbol{u}, \boldsymbol{u}$ and $\boldsymbol{v}$ so it is zero.

Property 3 The cross product of any vector with itself (two equal rows) is $\boldsymbol{u} \times \boldsymbol{u} = \boldsymbol{0}$.

When $\boldsymbol{u}$ and $\boldsymbol{v}$ are parallel, the cross product is zero. When $\boldsymbol{u}$ and $\boldsymbol{v}$ are perpendicular, the dot product is zero. One involves $\sin\theta$ and the other involves $\cos\theta$:

$$\|\boldsymbol{u} \times \boldsymbol{v}\| = \|\boldsymbol{u}\|\,\|\boldsymbol{v}\|\,|\sin\theta| \quad \text{and} \quad |\boldsymbol{u} \cdot \boldsymbol{v}| = \|\boldsymbol{u}\|\,\|\boldsymbol{v}\|\,|\cos\theta|. \quad (12)$$

Example 7 Since $\boldsymbol{u} = (3, 2, 0)$ and $\boldsymbol{v} = (1, 4, 0)$ are in the xy plane, $\boldsymbol{u} \times \boldsymbol{v}$ goes up the z axis:

$$\boldsymbol{u} \times \boldsymbol{v} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 3 & 2 & 0 \\ 1 & 4 & 0 \end{vmatrix} = 10\boldsymbol{k}. \quad \text{The cross product is } \boldsymbol{u} \times \boldsymbol{v} = (0, 0, 10).$$

The length of $u \times v$ equals the area of the parallelogram with sides u and v. This will be important: In this example the area is 10.

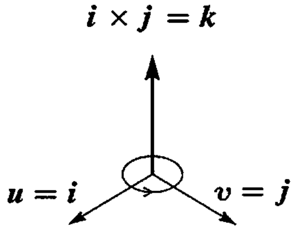
Example 8 The cross product of $\boldsymbol{u} = (1, 1, 1)$ and $\boldsymbol{v} = (1, 1, 2)$ is $(1, -1, 0)$:

$$\begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 1 & 1 & 1 \\ 1 & 1 & 2 \end{vmatrix} = \boldsymbol{i}\begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} - \boldsymbol{j}\begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} + \boldsymbol{k}\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} = \boldsymbol{i} - \boldsymbol{j}.$$

This vector $(1, -1, 0)$ is perpendicular to $(1, 1, 1)$ and $(1, 1, 2)$ as predicted. Area $= \sqrt{2}$.

Example 9 The cross product of $(1, 0, 0)$ and $(0, 1, 0)$ obeys the *right hand rule.* It goes up not down:

$$\begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = \boldsymbol{k}$$



Rule $\boldsymbol{u} \times \boldsymbol{v}$ points along your right thumb when the fingers curl from $\boldsymbol{u}$ to $\boldsymbol{v}$.

Thus $\boldsymbol{i} \times \boldsymbol{j} = \boldsymbol{k}$. The right hand rule also gives $\boldsymbol{j} \times \boldsymbol{k} = \boldsymbol{i}$ and $\boldsymbol{k} \times \boldsymbol{i} = \boldsymbol{j}$. Note the cyclic order. In the opposite order (anti-cyclic) the thumb is reversed and the cross product goes the other way: $\boldsymbol{k} \times \boldsymbol{j} = -\boldsymbol{i}$ and $\boldsymbol{i} \times \boldsymbol{k} = -\boldsymbol{j}$ and $\boldsymbol{j} \times \boldsymbol{i} = -\boldsymbol{k}$. You see the three plus signs and three minus signs from a 3 by 3 determinant.

The definition of $\boldsymbol{u} \times \boldsymbol{v}$ can be based on vectors instead of their components:

DEFINITION The ***cross product*** is a vector with length $\|\boldsymbol{u}\|\,\|\boldsymbol{v}\|\,|\sin\theta|$. Its direction is perpendicular to $\boldsymbol{u}$ and $\boldsymbol{v}$. It points "up" or "down" by the right hand rule.

This definition appeals to physicists, who hate to choose axes and coordinates. They see (u_1, u_2, u_3) as the position of a mass and (F_x, F_y, F_z) as a force acting on it. If $\boldsymbol{F}$ is parallel to $\boldsymbol{u}$, then $\boldsymbol{u} \times \boldsymbol{F} = \boldsymbol{0}$—there is no turning. The cross product $\boldsymbol{u} \times \boldsymbol{F}$ is the turning force or *torque*. It points along the turning axis (perpendicular to $\boldsymbol{u}$ and $\boldsymbol{F}$). Its length $\|\boldsymbol{u}\|\,\|\boldsymbol{F}\|\sin\theta$ measures the "moment" that produces turning.

Triple Product = Determinant = Volume

Since $\boldsymbol{u} \times \boldsymbol{v}$ is a vector, we can take its dot product with a third vector $\boldsymbol{w}$. That produces the ***triple product*** $(\boldsymbol{u} \times \boldsymbol{v}) \cdot \boldsymbol{w}$. It is called a "scalar" triple product, because it is a number. In fact it is a determinant—it gives the volume of the $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ box:

Triple product
$$(\boldsymbol{u} \times \boldsymbol{v}) \cdot \boldsymbol{w} = \begin{vmatrix} w_1 & w_2 & w_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}. \tag{13}$$

We can put $\boldsymbol{w}$ in the top or bottom row. The two determinants are the same because _____ row exchanges go from one to the other. Notice when this determinant is zero:

$$(\boldsymbol{u} \times \boldsymbol{v}) \cdot \boldsymbol{w} = 0 \quad \text{exactly when the vectors } \boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \text{ lie in the } \textit{same plane.}$$

First reason $\boldsymbol{u} \times \boldsymbol{v}$ is perpendicular to that plane so its dot product with $\boldsymbol{w}$ is zero.

Second reason Three vectors in a plane are dependent. The matrix is singular (det $= 0$).

Third reason Zero volume when the $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ box is squashed onto a plane.

It is remarkable that $(\boldsymbol{u} \times \boldsymbol{v}) \cdot \boldsymbol{w}$ equals the volume of the box with sides $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$. This 3 by 3 determinant carries tremendous information. Like $ad - bc$ for a 2 by 2 matrix, it separates invertible from singular. Chapter 6 will be looking for singular.

■ REVIEW OF THE KEY IDEAS ■

1. Cramer's Rule solves $A\boldsymbol{x} = \boldsymbol{b}$ by ratios like $x_1 = |B_1|/|A| = |\boldsymbol{b}\, \boldsymbol{a}_2 \cdots \boldsymbol{a}_n|/|A|$.
2. When C is the cofactor matrix for A, the inverse is $A^{-1} = C^{\mathrm{T}}/\det A$.
3. The volume of a box is $|\det A|$, when the box edges are the rows of A.
4. Area and volume are needed to change variables in double and triple integrals.
5. In $\mathbf{R}^3$, the cross product $\boldsymbol{u} \times \boldsymbol{v}$ is perpendicular to $\boldsymbol{u}$ and $\boldsymbol{v}$.

■ WORKED EXAMPLES ■

5.3 A If A is singular, the equation $AC^{\mathrm{T}} = (\det A)I$ becomes $AC^{\mathrm{T}} =$ **zero matrix.** *Then each column of C^{T} is in the nullspace of A.* Those columns contain cofactors along rows of A. So the cofactors quickly find the nullspace of a 3 by 3 matrix—my apologies that this comes so late!

Solve $A\boldsymbol{x} = \boldsymbol{0}$ by $\boldsymbol{x} =$ cofactors along a row, for these singular matrices of rank 2:

Cofactors give Nullspace
$$A = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 3 & 9 \\ 2 & 2 & 8 \end{bmatrix} \qquad A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

Any nonzero column of C^{T} will give the desired solution to $A\boldsymbol{x} = \boldsymbol{0}$. With rank 2, A has at least one nonzero cofactor. If A has rank 1 we get $\boldsymbol{x} = \boldsymbol{0}$ and the idea fails.

Solution The first matrix has these cofactors along its top row (note each minus sign):

$$\begin{vmatrix} 3 & 9 \\ 2 & 8 \end{vmatrix} = 6 \qquad -\begin{vmatrix} 2 & 9 \\ 2 & 8 \end{vmatrix} = 2 \qquad \begin{vmatrix} 2 & 3 \\ 2 & 2 \end{vmatrix} = -2$$

Then $\boldsymbol{x} = (6, 2, -2)$ solves $A\boldsymbol{x} = \mathbf{0}$. The cofactors along the second row are $(-18, -6, 6)$ which is just $-3\boldsymbol{x}$. This is also in the one-dimensional nullspace of A.

The second matrix has *zero cofactors* along its first row. The nullvector $\boldsymbol{x} = (0, 0, 0)$ is not interesting. The cofactors of row 2 give $\boldsymbol{x} = (1, -1, 0)$ which solves $A\boldsymbol{x} = \mathbf{0}$.

Every n by n matrix of rank $n-1$ has at least one nonzero cofactor by Problem 3.3.12. But for rank $n-2$, all cofactors are zero and we only find $\boldsymbol{x} = \mathbf{0}$.

5.3 B Use Cramer's Rule with ratios $\det B_j / \det A$ to solve $A\boldsymbol{x} = \boldsymbol{b}$. Also find the inverse matrix $A^{-1} = C^{\mathrm{T}} / \det A$. Why is the solution $\boldsymbol{x}$ for this $\boldsymbol{b}$ the same as column 3 of A^{-1}? Which cofactors are involved in computing that column $\boldsymbol{x}$?

$$A\boldsymbol{x} = \boldsymbol{b} \quad \text{is} \quad \begin{bmatrix} 2 & 6 & 2 \\ 1 & 4 & 2 \\ 5 & 9 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Find the volumes of the boxes whose edges are *columns* of A and then rows of A^{-1}.

Solution The determinants of the B_j (with right side $\boldsymbol{b}$ placed in column j) are

$$|B_1| = \begin{vmatrix} \mathbf{0} & 6 & 2 \\ \mathbf{0} & 4 & 2 \\ \mathbf{1} & 9 & 0 \end{vmatrix} = 4 \qquad |B_2| = \begin{vmatrix} 2 & \mathbf{0} & 2 \\ 1 & \mathbf{0} & 2 \\ 5 & \mathbf{1} & 0 \end{vmatrix} = -2 \qquad |B_3| = \begin{vmatrix} 2 & 6 & \mathbf{0} \\ 1 & 4 & \mathbf{0} \\ 5 & 9 & \mathbf{1} \end{vmatrix} = 2.$$

Those are cofactors C_{31}, C_{32}, C_{33} of row 3. Their dot product with row 3 is $\det A$:

$$\det A = a_{31}C_{31} + a_{32}C_{32} + a_{33}C_{33} = (5, 9, 0) \cdot (4, -2, 2) = 2.$$

The three ratios $\det B_j / \det A$ give the three components of $\boldsymbol{x} = (2, -1, 1)$. This $\boldsymbol{x}$ is the third column of A^{-1} because $\boldsymbol{b} = (0, 0, 1)$ is the third column of I. The cofactors along the other rows of A, divided by $\det A = 2$, give the other *columns* of A^{-1}:

$$A^{-1} = \frac{C^{\mathrm{T}}}{\det A} = \frac{1}{2} \begin{bmatrix} -18 & 18 & 4 \\ 10 & -10 & -2 \\ -11 & 12 & 2 \end{bmatrix}. \quad \text{Multiply to check} \quad AA^{-1} = I$$

The box from the columns of A has volume $= \det A = 2$ (the same as the box from the rows, since $|A^{\mathrm{T}}| = |A|$). The box from A^{-1} has volume $1/|A| = \frac{1}{2}$.

Problem Set 5.3

Problems 1–5 are about Cramer's Rule for $x = A^{-1}b$.

1 Solve these linear equations by Cramer's Rule $x_j = \det B_j / \det A$:

(a) $\begin{aligned} 2x_1 + 5x_2 &= 1 \\ x_1 + 4x_2 &= 2 \end{aligned}$ (b) $\begin{aligned} 2x_1 + x_2 &= 1 \\ x_1 + 2x_2 + x_3 &= 0 \\ x_2 + 2x_3 &= 0. \end{aligned}$

2 Use Cramer's Rule to solve for y (only). Call the 3 by 3 determinant D:

(a) $\begin{aligned} ax + by &= 1 \\ cx + dy &= 0 \end{aligned}$ (b) $\begin{aligned} ax + by + cz &= 1 \\ dx + ey + fz &= 0 \\ gx + hy + iz &= 0. \end{aligned}$

3 Cramer's Rule breaks down when $\det A = 0$. Example (a) has no solution while (b) has infinitely many. What are the ratios $x_j = \det B_j / \det A$ in these two cases?

(a) $\begin{aligned} 2x_1 + 3x_2 &= 1 \\ 4x_1 + 6x_2 &= 1 \end{aligned}$ (parallel lines) (b) $\begin{aligned} 2x_1 + 3x_2 &= 1 \\ 4x_1 + 6x_2 &= 2 \end{aligned}$ (same line)

4 *Quick proof of Cramer's rule.* The determinant is a linear function of column 1. It is zero if two columns are equal. When $\boldsymbol{b} = A\boldsymbol{x} = x_1\boldsymbol{a}_1 + x_2\boldsymbol{a}_2 + x_3\boldsymbol{a}_3$ goes into the first column of A, the determinant of this matrix B_1 is

$$|\boldsymbol{b} \quad \boldsymbol{a}_2 \quad \boldsymbol{a}_3| = |x_1\boldsymbol{a}_1 + x_2\boldsymbol{a}_2 + x_3\boldsymbol{a}_3 \quad \boldsymbol{a}_2 \quad \boldsymbol{a}_3| = x_1|\boldsymbol{a}_1 \quad \boldsymbol{a}_2 \quad \boldsymbol{a}_3| = x_1 \det A.$$

(a) What formula for x_1 comes from left side = right side?

(b) What steps lead to the middle equation?

5 If the right side $\boldsymbol{b}$ is the first column of A, solve the 3 by 3 system $A\boldsymbol{x} = \boldsymbol{b}$. How does each determinant in Cramer's Rule lead to this solution $\boldsymbol{x}$?

Problems 6–15 are about $A^{-1} = C^{\mathrm{T}} / \det A$. Remember to transpose C.

6 Find A^{-1} from the cofactor formula $C^{\mathrm{T}} / \det A$. Use symmetry in part (b).

(a) $A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 3 & 0 \\ 0 & 7 & 1 \end{bmatrix}$ (b) $A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}.$

7 If all the cofactors are zero, how do you know that A has no inverse? If none of the cofactors are zero, is A sure to be invertible?

8 Find the cofactors of A and multiply AC^{T} to find $\det A$:

$$A = \begin{bmatrix} 1 & 1 & 4 \\ 1 & 2 & 2 \\ 1 & 2 & 5 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 6 & -3 & 0 \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{bmatrix} \quad \text{and } AC^{\mathrm{T}} = \underline{\qquad}.$$

If you change that 4 to 100, why is $\det A$ unchanged?

9 Suppose $\det A = 1$ and you know all the cofactors in C. How can you find A?

10 From the formula $AC^{\mathrm{T}} = (\det A)I$ show that $\det C = (\det A)^{n-1}$.

11 If all entries of A are integers, and $\det A = 1$ or -1, prove that all entries of A^{-1} are integers. Give a 2 by 2 example with no zero entries.

12 If all entries of A and A^{-1} are integers, prove that $\det A = 1$ or -1. Hint: What is $\det A$ times $\det A^{-1}$?

13 Complete the calculation of A^{-1} by cofactors that was started in Example 5.

14 L is lower triangular and S is symmetric. Assume they are invertible:

To invert triangular L symmetric S
$$L = \begin{bmatrix} a & 0 & 0 \\ b & c & 0 \\ d & e & f \end{bmatrix} \qquad S = \begin{bmatrix} a & b & d \\ b & c & e \\ d & e & f \end{bmatrix}.$$

(a) Which three cofactors of L are zero? Then L^{-1} is also lower triangular.

(b) Which three pairs of cofactors of S are equal? Then S^{-1} is also symmetric.

(c) The cofactor matrix C of an orthogonal Q will be ____. *Why*?

15 For $n = 5$ the matrix C contains ____ cofactors. Each 4 by 4 cofactor contains ____ terms and each term needs ____ multiplications. Compare with $5^3 = 125$ for the Gauss-Jordan computation of A^{-1} in Section 2.4.

Problems 16–26 are about area and volume by determinants.

16 (a) Find the area of the parallelogram with edges $\boldsymbol{v} = (3,2)$ and $\boldsymbol{w} = (1,4)$.

(b) Find the area of the triangle with sides $\boldsymbol{v}$, $\boldsymbol{w}$, and $\boldsymbol{v} + \boldsymbol{w}$. Draw it.

(c) Find the area of the triangle with sides $\boldsymbol{v}$, $\boldsymbol{w}$, and $\boldsymbol{w} - \boldsymbol{v}$. Draw it.

17 A box has edges from $(0,0,0)$ to $(3,1,1)$ and $(1,3,1)$ and $(1,1,3)$. Find its volume. Also find the area of each parallelogram face using $\|\boldsymbol{u} \times \boldsymbol{v}\|$.

18 (a) The corners of a triangle are $(2,1)$ and $(3,4)$ and $(0,5)$. What is the area?

(b) Add a corner at $(-1,0)$ to make a lopsided region (four sides). Find the area.

19 The parallelogram with sides $(2,1)$ and $(2,3)$ has the same area as the parallelogram with sides $(2,2)$ and $(1,3)$. Find those areas from 2 by 2 determinants and say why they must be equal. (I can't see why from a picture. Please write to me if you do.)

20 The Hadamard matrix H has orthogonal rows. The box is a hypercube!

$$\text{What is} \quad |H| = \begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{vmatrix} = \text{volume of a hypercube in } \mathbf{R}^4?$$

21 If the columns of a 4 by 4 matrix have lengths L_1, L_2, L_3, L_4, what is the largest possible value for the determinant (based on volume)? If all entries of the matrix are 1 or -1, what are those lengths and the maximum determinant?

22 Show by a picture how a rectangle with area $x_1 y_2$ minus a rectangle with area $x_2 y_1$ produces the same area as our parallelogram.

23 When the edge vectors $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ are perpendicular, the volume of the box is $\|\boldsymbol{a}\|$ times $\|\boldsymbol{b}\|$ times $\|\boldsymbol{c}\|$. The matrix $A^{\mathrm{T}}A$ is _____. Find $\det A^{\mathrm{T}}A$ and $\det A$.

24 The box with edges $\boldsymbol{i}$ and $\boldsymbol{j}$ and $\boldsymbol{w} = 2\boldsymbol{i} + 3\boldsymbol{j} + 4\boldsymbol{k}$ has height _____. What is the volume? What is the matrix with this determinant? What is $\boldsymbol{i} \times \boldsymbol{j}$ and what is its dot product with $\boldsymbol{w}$?

25 An n-dimensional cube has how many corners? How many edges? How many $(n-1)$-dimensional faces? The cube in $\mathbf{R}^n$ whose edges are the rows of $2I$ has volume _____. A hypercube computer has parallel processors at the corners with connections along the edges.

26 The triangle with corners $(0,0), (1,0), (0,1)$ has area $\frac{1}{2}$. The pyramid in $\mathbf{R}^3$ with four corners $(0,0,0), (1,0,0), (0,1,0), (0,0,1)$ has volume _____. What is the volume of a pyramid in $\mathbf{R}^4$ with five corners at $(0,0,0,0)$ and the rows of I?

Problems 27–30 are about areas dA and volumes dV in calculus.

27 Polar coordinates satisfy $x = r\cos\theta$ and $y = r\sin\theta$. Polar area is $J\,dr\,d\theta$:

$$J = \begin{vmatrix} \partial x/\partial r & \partial x/\partial \theta \\ \partial y/\partial r & \partial y/\partial \theta \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}.$$

The two columns are orthogonal. Their lengths are _____. Thus $J =$ _____.

28 Spherical coordinates ρ, ϕ, θ satisfy $x = \rho\sin\phi\cos\theta$ and $y = \rho\sin\phi\sin\theta$ and $z = \rho\cos\phi$. Find the 3 by 3 matrix of partial derivatives: $\partial x/\partial\rho, \partial x/\partial\phi, \partial x/\partial\theta$ in row 1. Simplify its determinant to $J = \rho^2\sin\phi$. Then dV in spherical coordinates is $\rho^2 \sin\phi\, d\rho\, d\phi d\theta$, the volume of an infinitesimal "coordinate box".

29 The matrix that connects r, θ to x, y is in Problem 27. Invert that 2 by 2 matrix:

$$J^{-1} = \begin{vmatrix} \partial r/\partial x & \partial r/\partial y \\ \partial\theta/\partial x & \partial\theta/\partial y \end{vmatrix} = \begin{vmatrix} \cos\theta & ? \\ ? & ? \end{vmatrix} = ?$$

It is surprising that $\partial r/\partial x = \partial x/\partial r$ (**Calculus**, Gilbert Strang, p. 501). Multiplying the matrices J and J^{-1} gives the chain rule $\frac{\partial x}{\partial x} = \frac{\partial x}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial x}{\partial\theta}\frac{\partial\theta}{\partial x} = 1$.

30 The triangle with corners $(0,0)$, $(6,0)$, and $(1,4)$ has area ______. When you rotate it by $\theta = 60°$ the area is ______. The determinant of the rotation matrix is

$$J = \begin{vmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{vmatrix} = \begin{vmatrix} \frac{1}{2} & ? \\ ? & ? \end{vmatrix} = ?$$

Problems 31–38 are about the triple product $(u \times v) \cdot w$ in three dimensions.

31 A box has base area $\|u \times v\|$. Its perpendicular height is $\|w\| \cos\theta$. Base area times height = volume = $\|u \times v\| \, \|w\| \cos\theta$ which is $(u \times v) \cdot w$. Compute base area, height, and volume for $u = (2,4,0)$, $v = (-1,3,0)$, $w = (1,2,2)$.

32 The volume of the same box is given more directly by a 3 by 3 determinant. Evaluate that determinant.

33 Expand the 3 by 3 determinant in equation (13) in cofactors of its row u_1, u_2, u_3. This expansion is the dot product of u with the vector _____.

34 Which of the triple products $(u \times w) \cdot v$ and $(w \times u) \cdot v$ and $(v \times w) \cdot u$ are the same as $(u \times v) \cdot w$? Which orders of the rows u, v, w give the correct determinant?

35 Let $P = (1,0,-1)$ and $Q = (1,1,1)$ and $R = (2,2,1)$. Choose S so that $PQRS$ is a parallelogram and compute its area. Choose T, U, V so that $OPQRSTUV$ is a tilted box and compute its volume.

36 Suppose (x,y,z) and $(1,1,0)$ and $(1,2,1)$ lie on a plane through the origin. What determinant is zero? What equation does this give for the plane?

37 Suppose (x,y,z) is a linear combination of $(2,3,1)$ and $(1,2,3)$. What determinant is zero? What equation does this give for the plane of all combinations?

38 (a) Explain from volumes why $\det 2A = 2^n \det A$ for n by n matrices.

(b) For what size matrix is the false statement $\det A + \det A = \det(A + A)$ true?

Challenge Problems

39 If you know all 16 cofactors of a 4 by 4 invertible matrix A, how would you find A?

40 Suppose A is a 5 by 5 matrix. Its entries in row 1 multiply determinants (cofactors) in rows 2–5 to give the determinant. Can you guess a "Jacobi formula" for $\det A$ using 2 by 2 determinants from rows 1–2 *times* 3 by 3 determinants from rows 3–5?

Test your formula on the $-1, 2, -1$ tridiagonal matrix that has determinant = 6.

41 The 2 by 2 matrix AB =(2 by 3)(3 by 2) has a "Cauchy-Binet formula" for $\det AB$:

$$\det AB = \text{sum of (2 by 2 determinants in } A\text{) (2 by 2 determinants in } B\text{)}$$

(a) Guess which 2 by 2 determinants to use from A and B.

(b) Test your formula when the rows of A are $1,2,3$ and $1,4,7$ with $B = A^{\mathrm{T}}$.

Chapter 6

Eigenvalues and Eigenvectors

6.1 Introduction to Eigenvalues

Linear equations $Ax = b$ come from steady state problems. Eigenvalues have their greatest importance in *dynamic problems*. The solution of $du/dt = Au$ is changing with time—growing or decaying or oscillating. We can't find it by elimination. This chapter enters a new part of linear algebra, based on $Ax = \lambda x$. All matrices in this chapter are square.

A good model comes from the powers $A, A^2, A^3, \ldots$ of a matrix. Suppose you need the hundredth power A^{100}. The starting matrix A becomes unrecognizable after a few steps, and A^{100} is very close to $[\,.6\ \ .6;\ \ .4\ \ .4\,]$:

$$\underset{A}{\begin{bmatrix} .8 & .3 \\ .2 & .7 \end{bmatrix}} \quad \underset{A^2}{\begin{bmatrix} .70 & .45 \\ .30 & .55 \end{bmatrix}} \quad \underset{A^3}{\begin{bmatrix} .650 & .525 \\ .350 & .475 \end{bmatrix}} \quad \cdots \quad \underset{A^{100}}{\begin{bmatrix} .6000 & .6000 \\ .4000 & .4000 \end{bmatrix}}$$

A^{100} was found by using the *eigenvalues* of A, not by multiplying 100 matrices. Those eigenvalues (here they are 1 and $1/2$) are a new way to see into the heart of a matrix.

To explain eigenvalues, we first explain eigenvectors. Almost all vectors change direction, when they are multiplied by A. ***Certain exceptional vectors x are in the same direction as Ax. Those are the "eigenvectors"***. Multiply an eigenvector by A, and the vector Ax is a number λ times the original x.

The basic equation is $Ax = \lambda x$. The number λ is an eigenvalue of A.

The eigenvalue λ tells whether the special vector x is stretched or shrunk or reversed or left unchanged—when it is multiplied by A. We may find $\lambda = 2$ or $\frac{1}{2}$ or -1 or 1. The eigenvalue λ could be zero! Then $Ax = 0x$ means that this eigenvector x is in the nullspace.

If A is the identity matrix, every vector has $Ax = x$. All vectors are eigenvectors of I. All eigenvalues "lambda" are $\lambda = 1$. This is unusual to say the least. Most 2 by 2 matrices have *two* eigenvector directions and *two* eigenvalues. We will show that $\det(A - \lambda I) = 0$.